A3: RINGS AND MODULES, EXAMPLES SHEET 3

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Feedback to tom.sanders@maths.ox.ac.uk. I am particularly interested in the suitability of the sheet. My thanks to Richard Earl for many of the problems.

1. Show that the set $\mathbb{A}$ of elements in $\mathbb{C}$ that are algebraic over $\mathbb{Q}$ form a field. Show that $\mathbb{A}$ is *not* a finite extension of $\mathbb{Q}$.

[It may help to use the Tower Law in the first part, and consider the polynomials $X^n - 2$ *in the second.]*

2. Factorise the following into irreducible elements in the given rings.

(i) $36X^3 - 24X^2 - 18X + 12$ in $\mathbb{Z}[X]$;
(ii) $X^3 + Y^3$ in $\mathbb{Q}[X, Y]$;
(iii) $\pi^8 - \pi^4 - 2$ in $\mathbb{Q}[\pi]$;

3. Suppose that R is an integral domain and $x, y \in R$, and $r \in R^*$.

(i) (a) Show that the ideal $\langle x\rangle \cap \langle y\rangle$ is generated by l if and only if $x \mid l$ and $y \mid l$, and any $m \in R$ with $x \mid m$ and $y \mid m$ has $l \mid m$. We say that l is a lowest common multiple of x and y.
(b) Show that the ideal $\bigcap\{\langle d\rangle : d \in R, \langle x, y\rangle \subset \langle d\rangle\}$ is generated by g if and only if $g \mid x$ and $g \mid y$, and if $d \mid x$ and $d \mid y$ then $d \mid g$. We say that g is a greatest common divisor of x and y.

(ii) Write E for the unital subring of polynomials in $\mathbb{F}[X, Y]$ generated by $\{X^iY^j : i + j \equiv 0 \pmod 2\}$, so that E is an integral domain.
(a) Show that X^2 and XY have a lowest common multiple in $\mathbb{F}[X, Y]$.
(b) Show that X^2 and XY do not have a lowest common multiple in E.
(c) Show that X^2 and XY have a greatest common divisor in E.

(iii) (a) Show that if h is a greatest common divisor of x and y, and k is a greatest common divisor of rx and ry then $k \sim rh$.
(b) Show that x and y have a lowest common multiple if and only if rx and ry have a greatest common divisor for all $r \in R^*$.
(c) Show that every pair of elements in R has a greatest common divisor if and only if the intersection of every pair of principal ideals is principal.

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[This question may look less accessible than it is. If the sheet seems long then part (iii) could be dropped.]

4. Suppose that R is an integral domain with the ACCP in which every maximal ideal is principal. Show that R is a PID.

[Hint: It may help to suppose that I_0 is a non-principal ideal and create an ascending chain of ideals by dividing out any common factor of all the elements.]

5.

(i) Show that $\{6, 10, 15\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module but no proper subset does.
(ii) Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.
(iii) Show that $\mathbb{R}[X]/\langle X\rangle$ and $\mathbb{R}[X]/\langle X-1\rangle$ are isomorphic as rings but not as $\mathbb{R}[X]$-modules.

6. Write $\mathrm{Int}(\mathbb{Z})$ for the functions $\mathbb{N}_0 \to \mathbb{Z}$ arising as polynomials. Show that the maps

$$e_k : \mathbb{N}_0 \to \mathbb{Z}; x \mapsto \frac{x(x-1)\cdots(x-k+1)}{k!}$$

for $k \in \mathbb{N}_0$ (with e_0 identically 1) form a basis for $\mathrm{Int}(\mathbb{Z})$ so that $\mathrm{Int}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}[X]$ as a $\mathbb{Z}$-module. On the other hand show that $\mathrm{Int}(\mathbb{Z})$ and $\mathbb{Z}[X]$ are not isomorphic as rings.

[Hint: It may help to prove that if f has degree d then

$$f(x) = \sum_{k=0}^{d} \alpha_k e_k(x) \text{ where } \alpha_k = f(k) - \sum_{i=0}^{k-1} \alpha_i \binom{k}{i} \text{ for } k \in \mathbb{N}_0.$$

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