Graph Theory Part A **Problem Sheet 2** **Marc Lackenby**

1. Let G be a connected graph on $n \geq 3$ vertices. Suppose that for every pair of non-adjacent vertices x and y, $d(x) + d(y) \geq n$. Show how to find a Hamiltonian cycle in G efficiently.

2. For positive integers m and n, let $K_{m,n}$ be the bipartite graph with parts A and B having m and n vertices respectively, and where every vertex of A is joined to every vertex of B. For which m and n is $K_{m,n}$ Hamiltonian?

3. Show that any graph with n vertices and more than $(n(n-1)/2)-(n-2)$ edges is Hamiltonian. Find an example of a graph with n vertices and $(n(n-1)/2)-(n-2)$ edges that is not Hamiltonian.

4. Show that a graph is bipartite if and only if it does not have a cycle of odd length.

5. Show that any graph has a bipartite subgraph containing at least half of its edges.

6. Let G be a bipartite graph in which all vertices have the same nonzero degree. Show that G has a perfect matching.

7. Let G be a graph in which the degree of every vertex is in $\{1, \ldots, k\}$. Show that G has a matching with at least $|V(G)|/2k$ edges.

8. Let M be a square matrix. Say that M is doubly stochastic if it has non-negative entries, every row sum is 1 and every column sum is 1. Say that M is a permutation matrix if every row or column has one entry equal to 1 and all other entries equal to 0. Show that any doubly stochastic matrix M can be written as $M = \sum_{i=1}^{k} c_i P_i$ for some k, where P_i is a permutation matrix and $c_i > 0$ for $1 \leq i \leq k$.

9. A travelling salesman is required to follow a route that visits every city exactly once and returns to the starting point. How can (s)he find the best route? In the language of graph theory, we have a complete graph on n vertices, a positive weight $w(xy)$ 'the distance' assigned to each pair of vertices xy, and we are required to find a minimum weight 'Hamilton cycle', i.e. a cycle that uses all n vertices. We say that an algorithm is an r-approximation algorithm if it always finds some Hamilton cycle with weight that is at most r times as large as the minimum possible.

Suppose that you have an efficient computer program that implements an r-approximation algorithm for the travelling salesman problem (TSP). Describe how you could use this program to test whether any given graph contains a Hamilton cycle. *(It is known that the latter task is hard unless 'P=NP', so the conclusion of this exercise is that there is unlikely to be an efficient approximation algorithm for TSP.)*

10. *(continuation of the previous exercise)*

Show that the following algorithm gives a 2-approximation for TSP under the additional assumption that the distances satisfy the triangle inequality $w(xz) \leq w(xy) + w(yz)$ for all x, y, z.

Insertion Algorithm. Start with any vertex v_1. Suppose at step $1 \leq t < n$ we have a sequence of distinct vertices $s_t = (v_1, \ldots, v_t)$. Pick $1 \leq i \leq t$ and $u \notin \{v_1, \ldots, v_t\}$ to minimise $w(v_i u)$. Define s_{t+1} by inserting u into s_t after v_i. Output the Hamilton cycle defined by s_n.