# CALCULUS OF VARIATIONS

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## CONTENTS

## 1. INTRODUCTION: STATIONARY VALUES OF INTEGRALS

This course on the Calculus of Variations is a doorway to modern applied mathematics and theoretical physics, but also has various applications to pure mathematics. For examination purposes you can treat it as a comparatively self-contained and straightforward topic, but that is not its only purpose. The central point of the course is to show how abstract and non-obvious ideas can play a part in solving concrete problems. This development which will be taken much further in the Part B Classical Mechanics course.

As mathematics, this has a history in which the great figures of Euler, Lagrange, and Hamilton played a notable part in the 18th and 19th centuries. Although stimulated by physics, they created quite new ideas in mathematics which turned out to be vital in the 20th century formulation of quantum mechanics and relativity.

First, let us recall a basic principle from calculus about how we can look for maxima or minima of functions of one variable:

**Principle** (Maxima are stationary points)**.** *Let $f(x)$ be a 'nice' function. Then the maxima and minima of $f$ occur when $f'(x) = 0$.*

('Nice' can mean continuously differentiable in the principle above, but we will deliberately not try to classify precisely what sort of pathological exceptions might occur in this course, typically assuming that we are only interested in sufficiently smooth functions for us to not have to worry about justifying differentiation.)

The Calculus of Variations is a set of techniques where instead of looking for the maxima/minima of a function of one variable, you look at a functional (function of a function), and you want to choose the function which minimises some value. Such problems appear all the time in physics.

**Example 1** (Shortest path between two points)**.** *Given points $(x_1, y_1)$ and $(x_2, y_2)$, what is the shortest length path between them?*

*By rotating the plane (which doesn't affect distances) we may assume that $y_1 = y_2$.*

*Let us imagine that the path is given by a suitably nice (continuously differentiable) function $y(x)$. Then the length of the path is given by*

$$I[y] = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx,$$

*and the only constraints that we have on $y(x)$ is that $y(x_1) = y(x_2) = y_1$. Since $y'(x)^2 \geq 0$, we see that the integral is minimised when*

$$y'(x) = 0,$$

*so the straight line path minimises the distance.*

(It is intuitively obvious that any path can be approximated by a smooth path arbitrarily well, and so it suffices to just consider smooth paths; again, this is not the point of this course, and so we will often assume that we can restrict ourselves to sufficiently well-behaved functions without rigorous justification.)

Notice that we have deduced a *local* rule about what happens at one point from a *global* criterion — variation over *all possible paths*. This is the basic idea of variational calculus that we shall generalise considerably and apply to a wide range of problems.

One of the simplest ideas in physics is that light travels in straight lines. This observation gains much greater power when put in the following way: light travels in a straight line *because a straight line is the shortest distance between two points.* This may sound a trivial reformulation but it remains one of the basic ideas in Einstein's general theory of relativity and is strongly bound up with the modern understanding of light in terms of quantum electrodynamics. So it should be taken seriously!

**Example 2** (Biatholon problem)**.** *A person is travelling from point A to point B. They first need to cross a field (running at speed $c_1$) and then cross a river (swimming at speed $c_2$). What path should they follow to get from A to B fastest?*
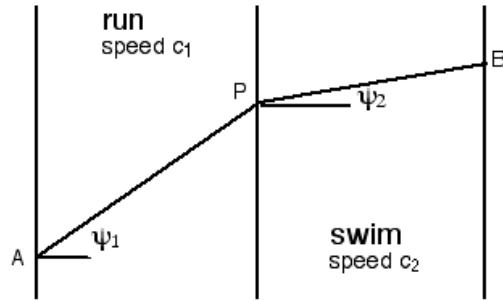


FIGURE 1. The biathlon problem

*Clearly the biathlete should run in as straight line from $A$ to $P$ on the edge of the river, and then swim in a straight line from $P$ to $B$, since straight lines minimise distance. (If $c_1 \geq c_2$ then consider the last point $P$ at which the athlete is on the field; the fastest way to get from $A$ to $P$ is running in a straight line in the field. After this point they must just swim from $P$ to $B$, and the fastest route is to swim in a straight line.)*

*Let $A = (x_A, y_A)$, $B = (x_B, y_B)$ and $P = (x_P, y_P)$. We therefore want to choose $y_P$ to minimise the total time. At the optimum value of $y_P$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}y_P} \left( \frac{\sqrt{(x_A - x_P)^2 + (y_A - y_P)^2}}{c_1} + \frac{\sqrt{(x_B - x_P)^2 + (y_B - y_P)^2}}{c_2} \right) = 0,$$

*so that*

$$\frac{(y_A - y_P)}{c_1 \sqrt{(x_A - x_P)^2 + (y_A - y_P)^2}} = \frac{(y_P - y_B)}{c_2 \sqrt{(x_B - x_P)^2 + (y_B - y_P)^2}}.$$

*So the optimum position of $P$ is such that the angles $\psi_1, \psi_2$ satisfy:*

$$(1) \qquad\qquad \frac{\sin \psi_1}{c_1} = \frac{\sin \psi_2}{c_2}.$$

You may recognise (1) as *Snell's Law* governing the refraction of light in its passage from one medium to another, provided that the observed *refractive index* of the medium is identified with the inverse of speed. Fermat observed that Snell's Law follows from such a least-time principle, although it was not until the 20th century that such a principle could be understood in terms of quantum physics and relativity.

We can now solve a slightly more general problem. Suppose that someone is running on a muddy field $x > 0$ where speed is proportional to $c(x)$, where $c(x)$ is some smooth function depending only on the $x$ coordinate. Equivalently, we have an optical medium with a continuously varying refractive index proportional to $(c(x))^{-1}$. What then is the shortest-time path from one point to another?

We can consider this in the following way. Divide up the muddy field into strips of thickness $\delta x$, so that in the strip from $x$ to $x + \delta x$, the speed is a constant given by $c(x)$.

Then repeatedly applying Snell's law from equation (1), it must be true that

$$(2) \qquad\qquad \frac{\sin(\psi(x))}{c(x)} \text{ is a constant of the path.}$$

Now take the limit as $\delta x \to 0$, and this law will remain true.

1.1. **A bit of elementary calculus:** The angle $\psi = \psi(x)$ that the path makes to the $x$-axis is such that $\tan \psi = \frac{dy}{dx} = y'(x)$. We also have arc-length $s$ defined by $ds^2 = dx^2 + dy^2$. Putting these together, we have

$$\sin \psi = \frac{y'}{\sqrt{1 + y'^2}} = \frac{dy}{ds}, \quad \cos \psi = \frac{1}{\sqrt{1 + y'^2}} = \frac{dx}{ds}.$$

It is also useful to derive from these that

$$\kappa = \frac{d\psi}{ds} = \frac{y''}{(1 + y'^2)^{3/2}}$$

where $\kappa$ is the *curvature* of the path, defined in a way that is invariant under rotation of the the axes.

**Example 3** (Shortest path on a 'muddy field'). *As an example of special interest, take the case where $c(x)$ is linear in $x$, in fact suppose $c(x) = x$. Then we have*

$$(3) \qquad \frac{\sin \psi(x)}{x} = constant,$$

*So we can translate the statement of Snell's law into a statement that $y = y(x)$ is a solution of*

$$(4) \qquad \frac{y'}{\sqrt{1 + y'^2}} = Ax$$

*If $A = 0$ this gives the lines $y = constant$, and for $A \neq 0$, we obtain*

$$(5) \qquad x^2 + (y - y_0)^2 = A^{-2}$$

*i.e. the circles with centres on the line $x = 0$. This completely solves the problem of finding the runner's shortest-time path between any two points on the field. We shall return later to this remarkable geometrical fact.*

Clearly we could now consider the even more general problem that arises when $c = c(x, y)$. But this is left to the worksheet to explore. Instead, we will take a different point of view. We reformulate the problem we have been studying in the following much more general terms.

We will think of the time taken to cover the path as a *functional* of the path taken. That is, it is a function on the space of possible paths, which are themselves functions.

Specifically, in the problem we have been considering, we can define a *functional* $I[y]$ for functions $y = y(x)$ by:

$$(6) \qquad I[y] = \int_a^b \frac{\sqrt{1 + y'(x)^2}}{c(x)} \, dx,$$

and then we ask for the least value of $I[y]$ as $y(x)$ varies over all possible paths. The function $y(x)$ which achieves this least value is called a *extremal*.

In this case it is obvious that we are looking at *minimum* values of an integral, but in general this is too restrictive. We use the term *stationary value*. This will mean that (in a sense to be defined) the first derivative of $I[y]$ vanishes. It will allow for a range of possibilities (a minimum, or maximum, or something equivalent to saddles, or more complicated situations in which higher derivatives also vanish).

We now regard this as a special case of a far more general problem in which we look for stationary values of

$$(7) \qquad\qquad I[y] = \int_a^b F(x, y(x), y'(x)) \, \mathrm{d}x \, .$$

for a fixed function $F(x_1, x_2, x_3)$. For simplicity we write this as $I[y] = \int_a^b F(x, y, y') \mathrm{d}x$.

The remarkable discovery (due principally to Euler and Lagrange) is that there is a *single method* which deals with all such questions. It can be extensively generalised further (to many dimensions, many derivatives, and constraints).

Even more remarkably, problems which don't look at all like least-time problems can usefully be reformulated in this way. Dynamical systems have trajectories which can be considered as being solutions to such an stationary-value problems, not of shortest distance or shortest time but of *least action,* as will be explained. One reason that this is a very useful description of physical problems is that the concept of the stationary value is independent of the coordinates used to describe it.

Theoretical physics today is rooted in the idea of stationary values of functionals of fields. The current Standard Model of particles and forces is defined by writing down a least action principle, as also are string and superstring theories. So part of the motivation for this course comes from the deepest properties of the physical world, properties which only come to light through the transforming power of creative mathematics.

## 2. The Euler-Lagrange equation

We now consider the general problem of finding the $y(x)$ which gives a stationary value to the functional

$$(8) \qquad I[y] = \int_a^b F(x, y, y') \, \mathrm{d}x \, .$$

From a completely rigorous point of view, we would have to specify the exact (huge) class of functions $y(x)$ over which the functional is taken (differentiable, differentiable with continuous derivative, differentiable to every order?), and we would also need some concept of what it means to vary a function to a 'nearby' function, by putting a metric or at least a topology on the class of functions.

In this course we will take a more elementary point of view and assume that all the functions we use have sufficient differentiability for the problem in hand. We will typically state results for smooth (infinitely differentiable) functions, since most situations in the real world are smooth (or can be approximated arbitrarily well by smooth functions). There is a Part C course which develops the more rigorous analysis. The one point that we will make rigorous, to help justify this rather cavalier approach, is the idea of a 'bump function'.

**Lemma 2.1** (Bump function). *There exists a function $B(x)$ with the following properties:*

(1) $B(x)$ *is infinitely differentiable,*

(2) $B(x) = 0$ *unless $x \in [0, 1]$,*

(3) $0 < B(x) \leq 1$ *if $x \in (0, 1)$.*

*Proof (sketch, non-examinable).* Let $B(x)$ be the function

$$B(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-1}(1-x)^{-1}), & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Then for all $n$, $B^{(n)}(x) \to 0$ as $x \downarrow 0$ or $x \uparrow 1$ (since the exponential decay dominates teh polynomial growth), so $f$ is infinitely differentiable at 0 and 1, so infinitely differentiable everywhere. Clearly $0 \leq B(x) \leq 1$ and $B(x) > 0$ iff $x \in (0, 1)$. $\quad \square$

By considering $B((x-a)/(b-a))$ we can define 'bump functions' on any interval $[a, b]$, and by scaling we can assume it takes the value 1 (its maximum) at the midpoint, which is sometimes convenient.

So a function can always be varied within any interval (by adding on a bump function) without affecting its differentiability, and it doesn't matter what degree of differentiability we are talking about.

**Lemma 2.2** (Test function lemma). *Let $y(x)$ be a continuous function on $[a, b]$ such that*

$$\int_a^b y(x)\eta(x)\mathrm{d}x = 0$$

*for every smooth function $\eta(x)$ with $\eta(a) = \eta(b) = 0$. Then $y(x) = 0$ for $x \in [a, b]$.*

*Proof.* For a proof by contradiction, suppose that $y(x_0) \neq 0$ for some $x_0 \in (a, b)$. W.L.O.G assume that $y(x_0) > 0$. Then we must have $y(x) > 0$ everywhere on some interval $[c, d]$ containing $x_0$ with $a < c < d < b$. (Since $y(x)$ is continuous.) Now take a bump function $b(x)$ on $[c, d]$. By assumption

$$\int_a^b y(x)b(x)\mathrm{d}x = 0,$$

but since $b(x) = 0$ unless $x \in [c, d]$ this means that

$$\int_c^d y(x)b(x)\mathrm{d}x = 0,$$

which is impossible since $y(x)b(x)$ is positive and continuous in this interval. This gives a contradiction, and so we must have that $y(x) = 0$ on $[a, b]$. $\qquad\square$

A small variation of this is the following.

**Lemma 2.3** (Test function lemma II). *Let $y(x)$ be a continuous function on $[a, b]$ and $c_1, c_2$ constants such that*

$$c_1\eta(a) + c_2\eta(b) + \int_a^b y(x)\eta(x)\mathrm{d}x = 0$$

*for every smooth function $\eta(x)$. Then $c_1 = c_2 = 0$ and $y(x) = 0$ for $x \in [a, b]$.*

*Proof.* First assume that $c_1 \neq 0$, so WLOG assume $c_1 > 0$. Let $\eta(x)$ be a bump function on the interval $[a - \epsilon, a + \epsilon]$, so that $\eta(a) = 1$, $\eta(x) = 0$ for $x > a + \epsilon$ and $\eta(x) \in [0, 1]$. We see that for $\epsilon$ small enough

$$0 = c_1\eta(a) + c_\eta(b) + \int_a^b y(x)\eta(x)dx \geq c_1 - \int_a^{a+\epsilon} |y(x)|dx > 0.$$

This gives a contradiction. Hence we must have $c_1 = 0$. An identical argument shows $c_2 = 0$. Then by Lemma 2.2 we have $y(x) = 0$. This gives the result. $\qquad\square$

Now we embark on the analysis of the stationary values of the functional $I(y)$. We might be tempted to try to vary $I[y]$ by some infinitesimal function $\delta(x)$, but there are uncountably many possible functions and this can lead to many difficulties. To avoid needing to worry about these many possibilities, we instead to focus on a single one-dimensional family of variations. We fix a function $\eta(x)$, and consider

$$(9) \qquad\qquad y(x) + \alpha\eta(x),$$

where $\alpha$ is a real parameter. This allows us to consider

$$(10) \qquad\qquad I[y + \alpha\eta] = \int_a^b F(x, y + \alpha\eta, y' + \alpha\eta')\, \mathrm{d}x.$$

Specifically, we have the following lemma.

**Lemma 2.4** (Minimisers give stationary values)**.** *Let $y(x)$ be a minimiser for $I[y]$, and $\eta(x)$ a smooth function. Then we have that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\right|_{\alpha=0} = 0.$$

*Proof.* This is just the standard calculus criterion for minima in disguise. If $y(x)$ is a minimiser of $I$ then $I[y + \alpha\eta] \geq I[y]$ for all $\alpha$ in the neighbourhood of zero, so $f(\alpha) = I[y + \alpha\eta]$ attains a minimum at $\alpha = 0$. Thus $f'(0) = 0$, which is the statement of the lemma. $\qquad\square$

**Lemma 2.5** (Constrained minimisers give stationary values)**.** *Let $y(x)$ be a minimiser for $I[y]$ subject to the constraint $y(a) = c_1$ and $y(b) = c_2$, and $\eta(x)$ a smooth function with $\eta(a) = \eta(b) = 0$. Then we have that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\right|_{\alpha=0} = 0.$$

*Proof.* This is the same as the previous proof, noting that if $\eta(a) = \eta(b) = 0$ then $y + \alpha\eta$ still satisfies the constraints $y(a) = c_1$ and $y(b) = c_2$. $\qquad\square$

**Theorem 2.6** (Euler-Lagrange equation for natural boundary condition)**.** *Let $I[y]$ be the functional*

$$I[y] := \int_a^b F(x, y, y')\, \mathrm{d}x$$

*for some smooth function $F$. Then the minimisers $y(x)$ of $I$ satisfy*

$$(11) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0,$$

*and*

$$(12) \qquad \frac{\partial}{\partial y'} F|_{x=a} = \frac{\partial}{\partial y'} F|_{x=b} = 0.$$

*Proof.* Let $y = y(x)$ be a minimiser of $I[y]$, and $\eta = \eta(x)$ a smooth function. By Lemma 2.4, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\Big|_{\alpha=0} = 0.$$

By applying the chain rule, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\Big|_{\alpha=0} = \int_a^b \left( \eta(x) \frac{\partial}{\partial y} F(x, y, y') + \eta'(x) \frac{\partial}{\partial y'} F(x, y, y') \right) \mathrm{d}x.$$

(Here by $\frac{\partial}{\partial y'} F(x, y, y')$ we mean $F_3(x, y, y')$, where the function $F_3$ is defined by $F_3(x, y, z) = \frac{\partial}{\partial z} F(x, y, z)$.)

The next key step is an integration by parts, to eliminate the $\eta'(x)$. First note that:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( \eta \frac{\partial}{\partial y'} F(x, y, y') \right) = \eta' \frac{\partial}{\partial y'} F(x, y, y') + \eta \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial}{\partial y'} F(x, y, y'),$$

so

$$\int_a^b \eta'(x) \frac{\partial}{\partial y'} F(x, y, y') \mathrm{d}x = \left[ \eta \frac{\partial}{\partial y'} F(x, y, y') \right]_a^b - \int_a^b \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial}{\partial y'} F(x, y, y') \mathrm{d}x.$$

(Here the $\frac{\mathrm{d}}{\mathrm{d}x}$ represents a *total* derivative, acting on every appearance of $x$ whether explicit or implicit (in $y$ and $y'$).)

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\Big|_{\alpha=0} = \left[ \eta \frac{\partial F}{\partial y'} \right]_a^b + \int_a^b \eta(x) \left( \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} \right) \mathrm{d}x.$$

Now, for $y$ to be an extremal, the LHS of this equation must vanish for *every* choice of $\eta$. Hence the RHS must vanish for all $\eta(x)$. But then Lemma 2.3 implies that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and

$$\frac{\partial}{\partial y'} F|_{x=a} = \frac{\partial}{\partial y'} F|_{x=b} = 0,$$

as required. $\qquad\qquad\square$

**Theorem 2.7** (Euler-Lagrange equation for fixed endpoint boundary condition)**.** *Let $I[y]$ be the functional*

$$I[y] = \int_a^b F(x, y, y') \, \mathrm{d}x$$

*for some smooth function $F$. Then the minimisers $y(x)$ of $I$ with $y(a) = c_1$ and $y(b) = c_2$ satisfy*

(13)
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0.$$

*Proof.* This is essentially the same as the previous proof, but by Lemma 2.5 we only consider functions $\eta$ satisfying $\eta(a) = \eta(b) = 0$. For all such functions we find that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha}I[y + \alpha\eta]\Big|_{\alpha=0} = \left[\eta\frac{\partial F}{\partial y'}\right]_a^b + \int_a^b \eta(x)\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)\mathrm{d}x$$

$$= \int_a^b \eta(x)\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)\mathrm{d}x.$$

(The term $[\eta\frac{\partial F}{\partial y'}]_a^b$ vanishes since $\eta(a) = \eta(b) = 0$. Now, by Lemma 2.2 we see that we must have

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0,$$

as required.                                                                          $\square$

Note: Remember that finding *extremals* and *stationary* values does not mean the same thing as locating maxima or minima. It will need some further piece of information to determine whether an extremal is a (local) maximum, or (local) minimum, or neither of these. However, maxima and minima must be extremals.

## 3. Classical examples and basic theorems

We first recall the two examples we saw before, now from the point of view of the Euler-Lagrange equations.

**Example 4** (Shortest distance on the Euclidean plane)**.** *Minimizing the distance of a path $y = y(x)$ between $(x_1, y_1)$ and $(x_2, y_2)$ is equivalent to minimizing $I[y] = \int_a^b F(x, y, y') \mathrm{d}x$ subject to $y(x_1) = y_1$, $y(x_2) = y_2$, where $F$ is given by*

$$(14) \qquad\qquad F(x, y, y') = \sqrt{1 + y'^2}.$$

*Since $\frac{\partial F}{\partial y} = 0$, the Euler-Lagrange equation becomes*

$$(15) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{y'}{\sqrt{1 + y'^2}} = 0.$$

*Hence $\frac{y'}{\sqrt{1+y'^2}}$ is constant, hence $y'$ is constant, and so we have a straight line.*

**Example 5** (Shortest paths on the 'muddy field')**.** *Next, we can verify the circular paths found for the 'muddy field' problem in lecture 1. We now take*

$$(16) \qquad\qquad F(x, y, y') = \frac{\sqrt{1 + y'^2}}{x}.$$

*The Euler-Lagrange equation is*

$$(17) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{y'}{x\sqrt{1 + y'^2}} = \frac{\partial F}{\partial y} = 0\,,$$

*and this immediately allows one integral to be done, leaving*

$$(18) \qquad\qquad \frac{y'}{x\sqrt{1 + y'^2}} = c\,,$$

*which is just the same equation (4) as we derived by generalizing Snell's Law. To remind you, the solutions are (arcs of) circles with centre on the y-axis. (Again, there are both fixed point and natural boundary conditions to consider, and you can check that these all give solutions which make sense.)*

3.1. **An 'ignorable coordinate'.** You should take particular note of the way that these problems simplified from a second-order ODE to a first-order ODE because this particular $F(x, y, y')$ had no explicit dependence on $y$, i.e. $\frac{\partial F}{\partial y} = 0$. This turns out to be of enormous importance, especially in applications to mathematical physics. The dependent variable $y$ is said to be *ignorable* in this situation. We can state a general theorem:

**Theorem 3.1** (Special case of Euler-Lagrange equation)**.** *Let $F(x_1, x_2, x_3)$ be a smooth function such that*

$$\frac{\partial}{\partial x_2} F(x_1, x_2, x_3) = 0.$$

*Let $y = y(x)$ be a minimiser for the functional*

$$I[y] := \int_a^b F(x, y, y') \mathrm{d}x.$$

*Then $\frac{\partial}{\partial y'} F(x, y, y')$ is a constant.*

3.2. **The same problem from a different standpoint.** If we consider the problem of finding stationary values of the functional $I[y]$ which comes from taking

$$(19) \qquad F(x, y, y') = \frac{\sqrt{1 + y'^2}}{y},$$

the geometrical interpretation tells us immediately that the extremals must be (arcs of) circles with centre on the $x$-axis. However, this is not immediately obvious if we write down the Euler-Lagrange equations:

$$(20) \qquad \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{y'}{y\sqrt{1 + y'^2}} \right) + \frac{\sqrt{1 + y'^2}}{y^2} = 0,$$

giving a complicated-looking second-order ODE. The key thing is to note a more general result which obtains when the $F$ has no explicit dependence on the $x$. This is *Beltrami's identity*, and is also of great importance.

**Theorem 3.2** (Beltrami's identity)**.** *Let $F(x_1, x_2, x_3)$ be a smooth function such that*

$$\frac{\partial}{\partial x_1} F(x_1, x_2, x_3) = 0.$$

*Let $y = y(x)$ be a minimiser of $I[y] = \int_a^b F(x, y, y') \mathrm{d}x$. Then we have*

$$(21) \qquad \frac{\mathrm{d}}{\mathrm{d}x} \left( y' \frac{\partial F}{\partial y'} - F \right) = 0,$$

*and so*

$$(22) \qquad H = y' \frac{\partial F}{\partial y'} - F = \text{ constant}.$$

*Proof.* Since $\partial F(x_1, x_2, x_3)/\partial x_1 = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y, y') = 0 + y' \frac{\partial}{\partial y} F(x, y, y') + y'' \frac{\partial}{\partial y'} F(x, y, y').$$

But by the Euler-Lagrange equation, this is

$$y' \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial}{\partial y'} F(x, y, y') + y'' \frac{\partial}{\partial y'} F(x, y, y') = \frac{\mathrm{d}}{\mathrm{d}x} \left( y' \frac{\partial}{\partial y'} F(x, y, y') \right),$$

which proves the result claimed.                                                    □

*Alternative proof.* Although the preceding proof is easy, it does not give any idea of why this first integral should exist. The following argument shows the reason: it is really just a special case of an ignorable coordinate. We simply exchange the roles of $x$ and $y$ and think of the curve to be found as a function $x(y)$ instead of as a function $y(x)$. (This is a clearly a very natural idea in the particular problem we are studying!) Writing $x'$ for $\mathrm{d}x/\mathrm{d}y$, so that $y' = (x')^{-1}$, the integral

$$(23) \qquad \int_a^b F(x, y, y') \,\mathrm{d}x, \quad y(a) = c, \ y(b) = d$$

becomes

$$(24) \qquad \int_c^d F(x, y, (x')^{-1}) x' \,\mathrm{d}y, \quad x(c) = a, \ x(d) = b$$

Now $x$ is the ignorable coordinate, so the Euler-Lagrange equation becomes

$$\frac{\partial}{\partial x'} \left( F(x, y, (x')^{-1}) x' \right) = \text{ constant.}$$

Taking care over the partial derivatives here, i.e. remembering how expressions like $\partial/\partial y' \, F(x, y, y')$ are properly defined, this yields

$$-(x')^{-2} F_3(x, y, (x')^{-1}) x' + F(x, y, (x')^{-1}) = \text{ constant.}$$

(Here $F_3(x_1, x_2, x_3) = \partial F(x_1, x_2, x_3)/\partial x_3$.) Thus

$$-y' F_3(x, y, y') + F(x, y, y') = \text{ constant.}$$

which is equivalent to the Beltrami identity.                                       □

Applied to the 'muddy field' problem, we deduce that

$$(25) \qquad H = \frac{-1}{y\sqrt{1 + y'^2}} \text{ is constant,}$$

and it is straightforward to perform the remaining integral and recover the circular paths.

In this case, however, there are no solutions satisfying the natural boundary conditions. This agrees with the fact that there is no minimum or maximum value for the integral between $x = a$ and $x = b$. It can take any real positive value, and the infimum 0 cannot be attained.

We shall come back to such shortest-path problems, or more generally the problems of *geodesics*, in Lecture 5. It will turn out that the 'muddy field' is actually a way of representing the core mathematical concept of the *hyperbolic plane*.

**Example 6** (Brachistochrone). *Find the curve which allows a smoothly falling particle released from rest at one point to reach a given lower point, not immediately below it, in the shortest time.*

*(We assume that gravity is a constant force g.) This is the most famous example of a stationary integral problem, originally solved by Newton, J. Bernoulli and others in the 17th century. (See $http: // mathworld. wolfram. com/ Brachistochrone Problem. html$). The answer is not at all intuitive.*

*This needs some first-year mechanics to obtain the relevant $F(x, y, y')$. In this problem we use x for horizontal distance and y for distance moved* downwards. *(This is purely for the sake of being able to start at the origin and yet avoid expressions like $\sqrt{-y}$.)*

*Explicitly, suppose the particle is released from $(x, y) = (0, 0)$ at $t = 0$, and then follows a curve $y = y(x)$ which reaches $(x, y) = (a, h)$, so that h is the height lost, and a the horizontal distance traversed. Using the initial conditions, and conservation of energy, we know that at each point in the motion along the curve $y = y(x)$,*

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = 0$$

*So*

$$\dot{x}^2 = \frac{2gy}{1 + y'^2}$$

*where $y' = \mathrm{d}y/\mathrm{d}x$, and so*

$$\mathrm{d}t = \frac{1}{\sqrt{2g}} \frac{\sqrt{1 + y'^2}}{\sqrt{y}} \mathrm{d}x\,,$$

*and hence the total time T is given, as a functional of the curve $y(x)$, by*

$$(26) \qquad\qquad T[y] = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,\mathrm{d}x\,.$$

*We want the curve $y(x)$ which minimises $T[y]$, subject to the fixed-end boundary conditions of passing through $(0, 0)$ and $(a, h)$. (Note that this can also be interpreted as solving the quickest path problem for the 'muddy field' where speed is proportional to $\sqrt{y}$.)*

*We could easily write down the Euler-Lagrange equations, but it's more efficient to take a short cut and use the Beltrami identity since $F(x, y, y')$ depends only on*

$y, y'$. *This tells us that*

(27)
$$\sqrt{y}\,\sqrt{1 + y'^2} = \sqrt{2c}$$

*for some constant $2c$. To solve, make the substitution $y = 2c\sin^2(\phi/2)$, and it becomes*

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = 2c\sin^2(\phi/2) = c(1 - \cos\phi)\,,$$

*and hence (using the initial condition)*

(28)
$$x = c(\phi - \sin\phi),\; y = c(1 - \cos\phi)\,,$$

*which is a **cycloid**. (See $\mathtt{http://mathworld.\,wolfram.\,com/\,Cycloid.\,html}$ for pictures).*

The ratio of $a$ to $h$ fixes the arc of the cycloid that solves the problem. If $a/h = \pi/2$, the cycloid is followed to its lowest point, at $\phi = \pi$, with $c = a/\pi$; if $a/h < \pi/2$ then it is a smaller segment of the cycloid, with $c$ chosen to fit, and so on.

It is worth filling in some more details. One finds that $\dot{\phi}$ is constant, namely $\sqrt{g/c}$. So the time taken to reach the point with parameter $\phi$ is just $\sqrt{c/g}\,\phi$. Suppose the horizontal distance $a$ is given, and we ask for the path which reaches it fastest, over all possible $h$. The time is given by $\sqrt{c/g}\,\phi$, where $c$ is given implicitly by the relation $a = c(\phi - \sin\phi)$. So finding the fastest way of reaching $a$ is equivalent to minimising $\frac{\phi}{\sqrt{\phi - \sin\phi}}$. One may check that this is given by $\phi = \pi$. This verifies what we obtain much more easily from taking the natural boundary condition $y' = 0$ at $x = a$. This selects the cycloid which arrives at $x = a$ at its lowest point, i.e. where $\phi = \pi$.

**Example 7** (Soap Film). *Consider a surface obtained by revolving the curve $y = y(x)$ around the $x$-axis, between the values $x = x_1$ and $x = x_2$. What curve gives the minimum area?*

*In this question the problem is to find a minimum* area, *but as it is the area of a surface of revolution, this reduces to finding a curve. This can be visualised as a soap film suspended between two circular wires at $x_1, x_2$, given that the film will establish an equilibrium at a position of minimum area.*

*In this case the functional $A[y]$ to be minimised is readily given as*

(29)
$$A[y] = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\,\mathrm{d}x\,.$$

*Again the Beltrami identity applies to gives us a first integral:*

$$\frac{y}{\sqrt{1 + y'^2}} = c$$

*of which the solutions are*

(30)
$$y = c \cosh\left(\frac{x - x_0}{c}\right).$$

Filling in the details and then fitting the initial conditions is a rather fiddly business and is left as an exercise.

The cosh curve will turn up again in connection with another problem — finding the shape taken by a hanging chain. It is called the *catenary* because of this connection, and the surface we have discovered is the *catenoid*. It plays a major part in the geometry of surfaces.

**Example 8** (A typical second order ODE problem). *Suppose*

(31)
$$F(x, y, y') = \frac{1}{2}y'^2 - \frac{1}{2}y^2 + y\, f(x), \quad y(0) = 0 = y(1).$$

*Then $\frac{\partial F}{\partial y'} = y', \frac{\partial F}{\partial y} = -y + f$, and the the Euler-Lagrange equation is*

(32)
$$y'' + y - f(x) = 0.$$

*In this case we don't have any helping hand from an ignorable coordinate or Beltrami's identity. However, we recognise the second-order ODE as the type of equation studied intensively in the Differential Equations courses, with the boundary conditions which can be solved by a Green's function.*

*In this course we shall not pursue the solutions of such equations any further; actually, we are more interested in a different question. Can we translate the differential equations we have met before into a problem of finding extremals?*

## 4. EXTENSION TO MANY VARIABLES AND HAMILTON'S PRINCIPLE

In this section we explore the application of variational principles to Mechanics.

First we need a modest generalization to allow more than one dependent variable. For this it is convenient to change our notation, since in mechanics applications it is actually *time* that is the one independent variable, and the many dependent variables represent the spatial coordinates of the mechanical system. So we think first about $q(t)$ and $F(t, q, \dot{q})$ instead of $y(x)$ and $F(x, y, y')$, where $q$ is a typical spatial coordinate and $t$ is time. There is a reason for using $q$ rather than $x$ as the dependent variable; we do not want to be restricted to Cartesian coordinates as use of the letter $x$ might wrongly suggest. The variable $q$ might be angle or radial distance, for instance. We then make a generalization to $q_1(t), q_2(t), \ldots q_n(t)$ and functions $F(t, q_1, \ldots q_n, \dot{q}_1, \ldots, \dot{q}_n)$. Thus we consider stationary values of the functional

$$(33) \qquad I[q_1, \ldots, q_n] = \int_a^b F(t, q_1, \ldots q_n, \dot{q}_1, \ldots, \dot{q}_n)\mathrm{d}t \,.$$

**Theorem 4.1.** *Let $F$ be a smooth function, and*

$$I[q_1, \ldots, q_n] := \int_a^b F(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)\mathrm{d}t.$$

*Then the minimisers $q_1 = q_1(t), \ldots, q_n = q_n(t)$ of $I$ satisfy*

$$(34) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{q}_i} - \frac{\partial F}{\partial q_i} = 0, \; for \; i = 1, \ldots, n$$

*and the boundary conditions*

$$(35) \qquad \left[\frac{\partial F}{\partial \dot{q}_i}\right]_a^b, \; for \; i = 1, \ldots, n \,.$$

*The minimisers of $I$ subject to the constraints $q_i(a) = c_{1,i}$ and $q_i(b) = c_{2,i}$ satisfy (34) but not necessarily (35).*

*Proof sketch.* The method of finding these is the same as in the simplest case; we choose an index $i$ a test function $\eta_i$, and temporarily fix $q_j$ for $j \neq i$ but vary $q_i$ with $q_i(t) \to q_i(t) + \alpha \eta_i(t)$. Since we have temporarily fixed $q_j$ for $j \neq i$, the functional $I$ is precisely the form of the cases already considered. The Euler-Lagrange equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{q}_i} - \frac{\partial F}{\partial q_i} = 0.$$

with boundary condition

$$\left[\eta_i \frac{\partial F}{\partial \dot{q}_i}\right]_a^b = 0.$$

Doing this for each index $i$ then gives the result. $\qquad\qquad\square$

We have the important special cases (1) of an 'ignorable coordinate' that arises when some variable $q_i$ does not appear in $F$:

$$(36) \qquad \frac{\partial F}{\partial q_i} = 0 \text{ implies } \frac{\partial F}{\partial \dot{q}_i} \text{ is a constant.}$$

and (2) the generalisation of the Beltrami identity that arises when $F$ is independent of $t$ :

$$(37) \qquad \frac{\partial F}{\partial t} = 0 \text{ implies } H := \sum_1^n \dot{q}_i \frac{\partial F}{\partial \dot{q}_i} - F \text{ is a constant.}$$

4.1. **Hamilton's Principle.** The following statement sums up why Mechanics can be reformulated in terms of extremal problems and solved by the calculus of variations.

**Definition 1.** *A constraint in a mechanical system is called **workless** if there is no friction (the constraint does no work);*

*A constraint is called **holonomic** if it is of the form $\phi(\tilde{q}_i, t) = 0$, where the $\tilde{q}_i$ are some set of coordinates. Specifically, the constraints do not involve the velocities $\dot{\tilde{q}}_i$.*

*A force is called **conservative** if it is the gradient of a potential $V$.*

**Principle** (Hamilton's Principle)**.** *If a mechanical system is subject only to holonomic, workless constraints and all forces are conservative, then the motion according to Newton's laws is an extremal of the integral*

$$(38) \qquad I[q] = \int L(q_i, \dot{q}_i, t)\mathrm{d}t \,,$$

*where the coordinates $q_i$ are arbitrary but unconstrained, and $L = T - V = Kinetic Energy - Potential Energy of the system as expressed in those coordinates. $L$ is called the* Lagrangian.

This is Hamilton's Principle, also referred to as the *principle of least action*, where the integral $I[q]$ is called the *action*.

In this course, we shall take it as given, not proved, that it correctly encodes physical laws. (In the Part B Classical Mechanics course it will be shown that it is equivalent to Newton's laws.)

Note that $I[q]$ has the dimensions of energy $\times$ time. *Action* is a technical term for a physical quantity with these dimensions. It turns out to be the most fundamental physical quantity (and in particular Planck's constant is a quantum of action.)

**Example 9** (Motion in free space without any forces). *The simplest example is just given by taking $L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The Euler-Lagrange equations are just*

$$(39) \qquad\qquad \ddot{x} = \ddot{y} = \ddot{z} = 0 \,,$$

*i.e. Newton's laws of motion for a free particle.*

**Example 10** (Motion in free space subject to a conservative force with potential). *The next simplest example arises from $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\psi(x, y, z)$ for motion in free space subject only to a conservative force with potential $\psi$ (typically, Newtonian gravity.) The Euler-Lagrange equations then become*

$$\ddot{x} = -\frac{\partial \psi}{\partial x}, \ddot{y} = -\frac{\partial \psi}{\partial y}, \ddot{z} = -\frac{\partial \psi}{\partial z}.$$

The value of the reformulation as a stationary integral often emerges more clearly if we make a *change of coordinates*. For orbit problems, with $\psi = -k/r$, the use of Cartesian $x, y, z$ is correct but not very helpful. Since the Lagrangian formalism does not mind which coordinates we use, it is more convenient to use spherical polars instead.

**Example 11** (Orbit problems with potential $\psi \propto -1/r$). *In polar coordinates $(r, \phi, \theta)$ for motion in free space with potential $\psi = -k/r$ we have*

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) + \frac{km}{r} \,.$$

*The $\theta$-equation is:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) - r^2 \sin\theta\cos\theta\dot{\phi}^2 = 0 \,,$$

*which is solved by $\theta \equiv \pi/2$, i.e. by paths always in the equatorial plane. Restricting our attention to such paths, the remaining equations become*

$$\ddot{r} - r\dot{\phi}^2 + \frac{k}{r^2} = 0 \,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) = 0 \,,$$

*which we can recognise as the equations obtained by a longer argument in the Prelims treatment. The $\phi$-equation obviously integrates to*

$$r^2\dot{\phi} = h \,.$$

*It is very important to note that the simplicity of this step arises directly from the fact that $\phi$ never appears in $L$; it is an ignorable coordinate. So in the Lagrangian formulation, the conservation of angular momentum is an* immediate *consequence.*

*The energy conservation statement can be equally easily derived; it is the equivalent of the Beltrami identity. By the remarks above, the fact that $L$ has no explicit dependence on $t$ means that*

$$(40) \qquad\qquad H = \sum_{i}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

*remains constant along the path.*

*It is immediate to see from the original form of $L$ (before the specialisation to equatorial paths) that in this case $H$ is just $T + V$, i.e. total energy. For equatorial paths we reduce to*

$$(41) \qquad\qquad \frac{1}{2}(\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{k}{r} = E \,,$$

*and hence now we have reduced the whole problem to a single integration, with its well known conic solutions.*

The two simplifying theorems we have used, that of ignorable coordinates and Beltrami's identity, point to a deep feature of physical theory. There is a direct connection between the concepts of *symmetry* (i.e. invariance under a group of transformations) and *conservation laws.*

Independence of angle $\phi$ means that the action is invariant under $\phi \to \phi + \alpha$, and this fact is equivalent to the conservation of angular momentum. In a problem where $x$ is ignorable, i.e. the action is invariant under $x \to x + \alpha$, the corresponding momentum in the $x$-direction is conserved. And when $t$ can be replaced by $t + \alpha$, we have a conserved energy.

Notice that angle $\times$ angular momentum, length $\times$ momentum, and time $\times$ energy, all have the dimensions of action. This conjugacy becomes fundamental in quantum mechanics, and is the basis of the famous Heisenberg Uncertainty Principle.

The Euler-Lagrange equations must remain the same in form under change of coordinates, because the concept of being stationary doesn't depend on which coordinates are used to describe the question. On a technical level this means that we can go ahead with writing down $T$ and $V$ in any way we like, without any chain-rule transformation of variables.

We shall just look at a few examples to illustrate this simplicity.

## 5. More examples in physics and geometry

So far we have not made use of the new freedom to impose holonomic constraints.

A typical problem studied in Prelims is where a particle moves smoothly on a surface of revolution, say the paraboloid $az = x^2 + y^2$. Let's derive the equations of motion from Hamilton's Principle.

**Example 12** (Movement on paraboloid). *At any time the position of the particle may be given as $(\sqrt{az}\cos\theta, \sqrt{az}\sin\theta, z)$. That is, we have used the holonomic constraint provided by the smooth surface to eliminate one of the three spatial dimensions and reduce the space to that of two dimensions. Here we have used $z, \theta$ as the two $q_i$ needed, but in principle we could have used whatever we liked. It's a good idea, however, to use the angle $\theta$ as one of the two coordinates because then it turns out to be ignorable in $L$ and so gives rise to an easy first integral. Explicitly,*

$$L = T - V = \frac{1}{2}((1 + \frac{a}{4z})\dot{z}^2 + az\dot{\theta}^2) - gz$$

*and the fact that $\theta$ is ignorable implies $\dot{\theta} = h/z$ for some constant $h$. The fact that $L$ has no explicit dependence on $t$, and that it is quadratic in the velocities, gives the fact that $T + V$ is conserved. Thus all the facts in the Prelims treatment are immediately derived without any dotting and wedging of vectors to eliminate the reaction force.*

Prelims questions do sometimes ask for the reaction force (e.g. to determine when a particle will lose contact with a surface) and if this is needed then a further step is required to deduce it from the acceleration of the particle. But in many contexts we are not actually concerned with this force at all and nothing is lost by eliminating it from the analysis altogether.

**Example 13** (Rotating particle on a straight wire; C.3 Mod 2010). *A particle moves smoothly on a straight wire which is at angle $\beta$ to the vertical and rotates at angular velocity $\omega$. The Mods method involves considering the normal reaction force and eliminating it. Using Hamilton's Principle we can ignore the normal reaction and go straight to $L = T - V$. The particle is at $(z\tan\beta\cos\omega t, z\tan\beta\sin\omega t, z)$. So the K.E. is just $1/2\{(z\omega\tan\beta)^2 + (\dot{z}\sec\beta)^2\}$ and the P.E. is $gz$. There is just one Euler-Lagrange equation, giving the equation of motion immediately as*

$$\ddot{z} - \omega^2\sin^2\beta z = -g\cos^2\beta$$

*as asked for in the question. This Mods question also asked whether $E = T + V$ is conserved, which it is not. (Obviously — because work has to be done to keep the rod rotating at the constant angular velocity $\omega$.) The Lagrangian method does*

*better, by producing an H which is conserved, but is not equal to total energy, namely*

$$H = \dot{z}\partial L/\partial\dot{z} - L = \frac{1}{2}\{(\dot{z}\sec\beta)^2 - (z\omega\tan\beta)^2\} + gz.$$

*Notice that this non-conservation of $T + V$ follows directly from the fact that $T$ is not a quadratic in the velocities.*

Now we are free to consider more general problems which it would not be easy to solve by the methods used in first-year questions.

**Example 14** (Movement on a general surface without forces). *Suppose we have a particle moving on a quite general surface embedded in three dimensions. (In what follows, we shall assume this constraint of contact with the surface without worrying about how it could be physically realised without the particle ever losing contact. For a mental picture, you might consider a spacecraft whose exterior surface is in the form of a double layer; the particle moves between these two layers so that the normal reaction can point either inwards or outwards.)*

*Hamilton's Principle leads us immediately to a Lagrangian for this motion: it is simply the kinetic energy $T$ for motion constrained to lie on the surface. Explicitly, suppose the surface is parametrised by $(u, v)$, so that its points are specified by $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Then writing $L$ in terms of the coordinates $(u, v)$, we have:*

$$L = T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(E(u, v)\dot{u}^2 + 2F(u, v)\dot{u}\dot{v} + G(u, v)\dot{v}^2)$$

*where*

$$E(u, v) = \mathbf{x}_u.\mathbf{x}_u, F(u, v) = \mathbf{x}_u.\mathbf{x}_v, G(u, v) = \mathbf{x}_v.\mathbf{x}_v$$

*We can now write down the Euler-Lagrange equations, thus in principle determining the entire motion. In general these second-order differential equations for $u$ and $v$ will not be easy to solve, but a simplifying feature is that the path taken by the particle is a* geodesic *on the surface — a stationary value of arc-length.*

*To show this, note first that a Lagrangian $L$ of a purely 'kinetic energy' form, (i.e. quadratic in the velocities $\dot{q}_i$, and with no explicit dependence on $t$) has a special property: by the Beltrami identity the value of $L$ is itself a constant of the motion.*

*The kinetic energy is also positive-definite. Now if $f$ is some strictly increasing function on the positive reals, consider the stationary value problem generated by $f(L)$. The Euler-Lagrange equations will be*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f(L)}{\partial\dot{q}_i} - \frac{\partial f(L)}{\partial q_i} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f'(L)\frac{\partial L}{\partial\dot{q}_i}\right) - f'(L)\frac{\partial L}{\partial q_i} = 0$$

$$f''(L)\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + f'(L)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) = 0$$

but as $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$, and $f'(L) \neq 0$, this reduces to the same equations as generated by $L$.

Taking $f(L)$ to be $\sqrt{L}$, this tells us that

$$\int \sqrt{E(u,v)\dot{u}^2 + 2F(u,v)\dot{u}\dot{v} + G(u,v)\dot{v}^2}\,\mathrm{d}t$$

generates the same Euler-Lagrange equations. But this is simply the arc-length for a trajectory on the surface, defining a geodesic where it is stationary.

If we wish we can eliminate the time variable $t$ and write the integral as

$$\int \sqrt{E(u,v) + 2F(u,v)v_u + G(u,v)v_u^2}\,\mathrm{d}u$$

where now $v = v(u)$ is being considered as defining the curve on the surface. This is of the same form as we studied earlier.

So in the absence of forces, a particle simply takes the shortest path (at least in the sense of a local minimum) it can, consistent with geometrical constraints. In this case, least action actually coincides with shortest distance. This is a generalization of Newton's second law.

### 5.1. More geodesics.

**Example 15** (Circular cylinder). *Take the surface to be the circular cylinder of radius 1 and axis along the $z$-axis. It is then given by*

$$\mathbf{x}(u,v) = (\cos u, \sin u, v).$$

*We then calculate $\mathbf{x}_u = (-\sin u, \cos u, 0), \mathbf{x}_v = (0,0,1)$, so that $E = G = 1, F = 0$. The kinetic energy Lagrangian is just*

$$L(u,v,\dot{u},\dot{v}) = \frac{1}{2}(\dot{u}^2 + \dot{v}^2)$$

*and the geodesics are given by*

$$\ddot{u} = \ddot{v} = 0$$

*and so are straight lines in the $(u,v)$ coordinates. The same conclusion comes equally easily from finding the geodesic as stationary arc-length, where the method above gives $v_{uu} = 0$, i.e. $v = au + b$, as the equation of the geodesics. (Note that paths on the cylinder illustrate very clearly that a local minimum of path-length is not at all the same thing as the absolute minimum.)*

*Why is this so simple? The point is that although the cylinder has been given as a curved surface in $\mathbb{R}^3$, it is in fact* intrinsically flat, *as is intuitively obvious:*

*the surface can be unwrapped without any stretching and laid out on a Euclidean plane. The proper word for this is that it is* isometric *to the plane. Under such an isometry, the geodesics are unchanged, since they are defined intrinsically.*

A similar example (a circular cone) is left to the worksheet.

It is worth noting that the concept of geodesic on a surface is much more general than this. There is no need to restrict to surfaces as defined by an embedding in an ambient three-dimensional space. The metric can be given abstractly (in fact we did this with our 'speed' functions in the opening lecture). Also, there is no need to restrict attention to geodesics on surfaces; we could equally well study geodesics in spaces of any number of dimensions.

In physics, this is a most important idea in the development of Einstein's general theory of relativity. In this theory, gravity becomes a part of the four-dimensional space-time geometry, not a force, and the orbits of free fall under gravity (including light rays) must be geodesics in the resulting space; the four-dimensional space is not thought of as embedded in anything bigger.

In pure mathematics, the study of geodesics is a vital part of Geometry and something you could follow in the B course next year.

## 6. Generalization to several independent variables and to higher derivatives

6.1. **Several independent variables.** Suppose that instead of considering the stationary values of functionals of a curve $y(x)$, we go up one dimension and consider the variation of surfaces $z(x, y)$. Thus we define the functional

$$I[z] = \int \int_R F(x, y, z, z_x, z_y) \mathrm{d}x \mathrm{d}y \,,$$

where $R$ is some region in the $(x, y)$-plane, and $z_x, z_y$ are the partial derivatives of $z(x, y)$ with respect to $x$ and $y$.

For example, $F(x, y, z, z_x, z_y) = \sqrt{1 + z_x^2 + z_y^2}$ would give the area of the surface, and so allow the investigation of minimal surfaces in generality (not restricted to surfaces of revolution).

The method, as always, is to vary the dependent variable along a one-dimensional path:

$$z(x, y) \to z(x, y) + \alpha \eta(x, y) \,,$$

which means that

$$\left. \frac{\mathrm{d}I}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int \int_R \left( \eta \frac{\partial F}{\partial z} + \eta_x \frac{\partial F}{\partial z_x} + \eta_y \frac{\partial F}{\partial z_y} \right) \mathrm{d}x \mathrm{d}y \,.$$

We can integrate by parts. In the case of fixed boundary conditions, i.e. $\eta = 0$ on $\partial R$, we obtain:

$$\left. \frac{\mathrm{d}I}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int \int_R \eta \left( \frac{\partial F}{\partial z} - \frac{\partial}{\partial x} \frac{\partial F}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial z_y} \right) \mathrm{d}x \mathrm{d}y \,.$$

and conclude that the Euler-Lagrange equation, which must hold at all points in $R$, is

$$\frac{\partial}{\partial x} \frac{\partial F}{\partial z_x} + \frac{\partial}{\partial y} \frac{\partial F}{\partial z_y} - \frac{\partial F}{\partial z} = 0 \,.$$

Further generalization, to $n$ rather than 2 independent variables, is immediate. The result is the following theorem.

**Theorem 6.1.** *Let $F$ be a smooth function, and $I$ the functional*

$$I[u] = \int_R F(x_1, \ldots, x_n, u, u_1, \ldots, u_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

*over a region $R \subseteq \mathbb{R}^n$ for smooth functions $u = u(x_1, x_2 \ldots x_n)$ where we write $u_i$ for $\partial u / \partial x_i$. Then the minimisers of $I[u]$, subject to fixed boundary conditions on*

$\partial R$, satisfy the Euler-Lagrange equation

$$\sum_i^n \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_i} - \frac{\partial F}{\partial u} = 0 \,.$$

Note: In the above theorem we assume that not only $u$, but also the region $R$ is sufficiently 'nice' for the statement to hold.

**Example 16** (Reformulation of wave equation). *A simple and beautiful example of this is the case where*

$$F = \frac{1}{2}|\nabla u|^2 = \frac{1}{2} \sum_1^n u_i^2$$

*in which case the Euler-Lagrange equation is just*

$$0 = \sum_i^n \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_i} - \frac{\partial F}{\partial u} = \sum_i^n \frac{\partial}{\partial x_i} u_i = \nabla^2 u \,,$$

*i.e. the Laplace equation or its n-dimensional generalization. This indicates that Laplacian or wave-equation problems can readily be reformulated in a variational form — an idea which is fundamental to modern quantum field theory.*

6.2. **Higher derivatives.** Suppose now we wish to find stationary values for

$$I[y] = \int_a^b F(x, y, y', y'') \,\mathrm{d}x \,.$$

Varying $y(x)$ as before, we find

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha}\Big|_{\alpha=0} = \int_a^b \left( \eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} + \eta'' \frac{\partial F}{\partial y''} \right) \mathrm{d}x \,.$$

Integrating by parts *twice,* we find

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha}\Big|_{\alpha=0} = \left[ \eta(\frac{\partial F}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''}) + \eta'\frac{\partial F}{\partial y''} \right]_a^b$$
$$+ \int_a^b \eta \left( \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} \right) \mathrm{d}x \,.$$

Thus we now have, as necessary condition for a stationary solution, the satisfaction of the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} = 0 \,.$$

This is a fourth-order differential equation, requiring four constants of integration. These must come from a suitable selection of end-point conditions (now on both $y$ and $y'$), and natural boundary conditions

$$\frac{\partial F}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''} = 0, \quad \frac{\partial F}{\partial y''} = 0\,.$$

**Example 17** (A diving board). *We shall study a problem which gives a picture of how the calculus of variations can solve practical problems of optimisation such as arise in engineering and economics.*

*We consider the functional*

$$E[y] = \int_0^L (\frac{1}{2}K(y'')^2 + \rho g y)\,\mathrm{d}x\,,$$

*which can be considered as the total energy of an elastic beam of horizontal length $L$, clamped at $x = 0$ so that it has $y = 0, y' = 0$ there, but free at $x = L$ and bending under its weight. (We assume that $y$ is suitably small, so that this functional is a reasonable approximation to the physical situation.) The beam will settle in an equilibrium where the total energy is minimised, and so the calculus of variations gives a method to find the shape of the beam.*

*The Euler-Lagrange equation is*

$$Ky'''' + \rho g = 0$$

*and the four boundary conditions are supplied by $y(0) = y'(0) = 0$ at one end, and then the natural boundary conditions $y''(L) = y'''(L) = 0$ at the other. This clearly specifies a quartic polynomial, and satisfaction of the boundary conditions gives*

$$y(x) = -\frac{\rho g}{24K}(x^4 - 4Lx^3 + 6L^2x^2)\,.$$

*Note that in this situation, the free end of the board will droop to height $y = -\frac{\rho g L^4}{8K}$. Imagine a swimmer in the pool putting a hand to the free end and fixing it at height $y = -\frac{\rho g L^4}{8K} + h$. Clearly, if $h = 0$ no force is required at all. But for $h \neq 0$ a force will be required. We can evaluate this force by extending the analysis.*

*First, solve the stationary problem again but now for the fixed-end condition $y(L) = -\frac{\rho g L^4}{8K} + h$. To shorten the expressions, write $w = \frac{\rho g}{K}$ in what follows. We find, straightforwardly, that now*

$$y(x) = -\frac{w}{24}(x^4 - 4Lx^3 + 6L^2x^2) + \frac{h}{2L^4}(-Lx^3 + 3L^2x^2)\,.$$

*Clearly the energy functional $E[y]$ can now be considered as a function of $h$. It will take the least value when $h = 0$. If the free end is raised, the energy will increase,*

*and this can only come from the work done, which is given by $\int F(h)\mathrm{d}h$, where $F(h)$ is the force needed to keep $y(L) = -\frac{w}{8L^4} + h$. Thus $F(h) = \frac{\mathrm{d}}{\mathrm{d}h}E(h)$.*

*This is easily calculated and is $\frac{3hK}{L^3}$.*

*Returning to the situation where the end $x = L$ is free, we can apply the same ideas to find the forces being applied at $x = 0$ in order to maintain the constraints. In this case it is even easier to see that the upward force to maintain $y(0) = 0$ is just the total weight $\rho g L$ of the board; slightly less obvious is that a* torque *of moment $\frac{1}{2}\rho g L^2$ is applied by the clamp to maintain the condition $y'(0) = 0$. In this case we use torque times angle = work done.*

You will already be familiar with this idea of a *force* being associated with a constraint, since it is just the idea of a normal reaction that you had in Prelims mechanics.

But suppose the functional is something that measures not energy but *cost*. Then the elements of the problem, including the constraints, take on an *economic* interpretation. You could imagine this diving-board curve as representing the effect of a company buying a hospital and changing a policy of stable employment to one of running down the work force. (The independent variable $x$ is now time, and $y$ measures the size of the work force.) How can it pursue this policy at least cost? Suppose that its cost functional is given by the same elements as appeared in the diving-board functional: the wages, proportional to $y$, and the cost in administrative disruption, strikes, etc. from making swingeing cuts, modelled as proportional to $(y'')^2$. The solution with natural boundary conditions will represent the ideal situation (from the point of view of the company, of course, not that of the patients!) at the end of a period. If a government regulator imposes a constraint, of dictating what the workforce level must be at that point, that constraint is naturally associated with a *price:* it is what it will be worth the company paying to persuade the regulator to reduce the imposed quota by one unit. In the Optimisation course, using *linear programming*, you met the idea that prices are *dual variables* associated with constraints, and this is a another example of it.

## 7. Extremals subject to an integral constraint

The problem addressed in this section is that of how to find a stationary value of an integral

$$I[y] = \int_a^b F(x, y, y') \, \mathrm{d}x$$

subject to an integral constraint

$$J[y] = \int_a^b G(x, y, y') \, \mathrm{d}x = C \, .$$

Now recall the method of Lagrange multipliers from Prelims. If we seek a stationary value of $I[y]$, constrained by $J[y] = C$, we can look for stationary values of

$$I[y] + \lambda(J[y] - C)$$

over all choices of $\lambda$ and $y(x)$. This means that $y(x)$ is a stationary value of $I[y] - \lambda J[y]$ for some constant $\lambda$, and so $y$ must satisfy the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y'}(F - \lambda G)\right) - \frac{\partial}{\partial y}(F - \lambda G) = 0 \, ,$$

for some constant $\lambda$. Of course $y$ must also satisfy the corresponding fixed end point or natural boundary conditions, where the natural boundary conditions are now

$$\frac{\partial}{\partial y'}(F - \lambda G) = 0 \qquad \text{at } x = a \text{ and } x = b$$

Let us record this as a useful theorem.

**Theorem 7.1.** *Let $F, G$ be two smooth functions, and*

$$I[y] := \int_a^b F(x, y, y') \, \mathrm{d}x, \qquad J[y] := \int_a^b G(x, y, y') \, \mathrm{d}x.$$

*then any smooth stationary value of $I[y]$ subjet to the constraint $J[y] = C$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y'}(F - \lambda G)\right) - \frac{\partial}{\partial y}(F - \lambda G) = 0 \, ,$$

*for some constant $\lambda$.*

## A freely hanging chain — the catenary:

We can use this method to find the shape taken by an (idealised) hanging chain of constant density, supported only by two ends. Assume that the chain falls on a curve described by $y = y(x)$, with fixed endpoints $y = b$ at $x = \pm a$. It is subject to the constraint that its total length is fixed:

$$J[y] = \int_{-a}^{a} \sqrt{1 + y'^2}\, \mathrm{d}x = L\,,$$

and then its equilibrium is determined by minimising its gravitational potential energy, which is

$$I[y] = g\rho \int_{-a}^{a} y\, \sqrt{1 + y'^2}\, \mathrm{d}x\,.$$

Applying the Lagrange multiplier method, and absorbing $\rho g$ into the $\lambda$,

$$F - \lambda G = (y - \lambda)\sqrt{1 + y'^2}\,,$$

which has no explicit $x$-dependence, so Beltrami's identity gives a first integral:

$$(y - \lambda) = c\sqrt{1 + y'^2}$$

Substituting $y = \lambda + c \cosh u$ readily gives the solution:

$$y = \lambda + c \cosh(\frac{x - x_0}{c})$$

Fitting these constants $c, \lambda, x_0$ to the given data $a, b, L$ is left as an exercise.

## Dido's Problem:

Another classical problem of this nature is the simplest example of an *isoperimetric problem*. On the Euclidean plane, given a fixed length as a perimeter, what is the largest area that can be enclosed by it? (The answer is given by taking a circle.) We will consider a slightly different version of this problem, in which the area is on one side of a given straight line, w.l.o.g the $x$-axis. Then the answer is given by taking the boundary to be a circular arc. You will find this described as 'Dido's problem', since it can somewhat fancifully be considered as arising in the *Aeneid*. Dido (better known for inspiring Purcell's famous Lament) supposedly fixed the boundaries of Carthage by this criterion. That is, the line $y = 0$ is the Mediterranean coastline.
(See http://mathworld.wolfram.com/DidosProblem.html).

For this problem, we could take $F = y$ and $G = \sqrt{1 + y'^2}$, where the boundary curve is taken to be $y = y(x)$, but it is actually better to take the boundary curve

to be given by $(x(t), y(t))$, where $t$ is an arbitrary parameter. Then we consider the extremals of

$$\int (y\dot{x} - \lambda\sqrt{\dot{x}^2 + \dot{y}^2})\mathrm{d}t \,.$$

The Euler-Lagrange equations are

(42) $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \dot{x}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{-\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -\dot{y}\,,$$

so

(43) $$\frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = (x - a)\,, \quad \frac{-\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -(y - b)\,,$$

and thus, eliminating $\lambda$,

$$(x - a)\dot{x} + (y - b)\dot{y} = 0, \quad (x - a)^2 + (y - b)^2 = c^2\,,$$

so the curves must be circles.

For the original Dido problem we are interested in the case of a *fixed* boundary condition $y(t) = 0$ at each end, and a *natural* boundary condition for $x(t)$ at each end (that is, we are taking the extremals over all possible $x$, given $y = 0$.) The natural boundary condition for $x$ is $y - \lambda\dot{x}/\sqrt{\dot{x}^2 + \dot{y}^2} = 0$, and since $y = 0$, this means $\dot{x} = 0$. (This means that $\mathrm{d}y/\mathrm{d}x$ is infinite here, which is why the $y(x)$ formulation is not appropriate.) This means that the centre of the circle must be on $y = 0$, and the stationary area, given the constraint, will be bounded by a semi-circle, as expected.

These are the same semi-circles as we have met in the problem of the quickest paths on the muddy field (more formally, geodesics on the hyperbolic plane). This can be seen more directly if we proceed slightly differently. The second equation in (42) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - y\Big) = 0, \quad \text{so} \quad \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - y \text{ is constant.}$$

When the boundary conditions $y = 0, \dot{x} = 0$ are imposed, this constant must be 0, and so

$$y\sqrt{1 + y'^2} = \lambda\,.$$

This is the same equation as arises for the quickest-path problem, at (25), and so has the same semi-circle solutions.

This feature extends to the more general colonial land-grab problem where a varying value $h(y)$ is attached to the land and the objective is to secure the greatest total value, given the length of the border to be defended. In this case, the problem

is given by taking $F = H(y)$ and $G = \sqrt{1 + y'^2}$, where $H(y) = \int_0^y h(u)\mathrm{d}u$. For the case of natural boundary conditions, the resulting equation is

$$H(y)\sqrt{1 + y'^2} = \lambda\,,$$

which you can check is the same equation as arises from the problem of finding the quickest path on the muddy field when the speed of movement is $H(y)$.

Thus if $h(y) = 1/\sqrt{y}$, so $H(y) = 2\sqrt{y}$, and the solution is given by a cycloid, just as with the Brachistochrone (see equation (27)). The details are left to the worksheet.

## Constraints and prices again

We now have another example of where a constraint can be thought of as defining a *price*. How much more $I$ can we get if we change the constraint $J = C$ to $J = C + \delta C$? Write $I(C)$ for the stationary value of $I$, given the constraint $J = C$. Then we find that

$$(44) \qquad\qquad\qquad \lambda = \frac{\mathrm{d}I(C)}{\mathrm{d}C}\,,$$

giving a nice interpretation of the $\lambda$.

For a proof of this, recall that the solution $I - \lambda J$ is stationary, i.e. is unchanged, to first order, when the extremal $y$ is changed to any $y + \delta y$ consistent with the boundary conditions. Suppose we choose the particular $\delta y$ which makes $y + \delta y$ the extremal for the problem where the constraint $J = C + \delta C$ is imposed. Then we have

$$I(C) - \lambda C = I(C + \delta C) - \lambda(C + \delta C)$$

to first order. Subtracting and taking $\delta C \to 0$, we recover the relation.

Thus, in Dido's problem the value of $\lambda$ in the solution indicates the value (in extra area gained) of increasing the length of the rope which defines the perimeter. (So we have solved an extra problem: how much money Dido should pay for old rope.)

Specifically, in this problem, we have for perimeter of length $L$, a stationary area $I(L) = \frac{L^2}{2\pi}$, and so $\frac{\mathrm{d}I(L)}{\mathrm{d}L} = \frac{L}{\pi}$, which is just the radius of the circle. It is easy to check that this is indeed the value of $\lambda$.

## 8. APPLICATION TO STURM-LIOUVILLE EQUATIONS

8.1. **Some motivation from quantum mechanics.** In quantum mechanics, the state of a physical system depends not on the motion of point particles but of *wave functions*. These are actually complex-valued, but for the purpose of this discussion we can give a simplified picture with real functions. The simplest situation is for a single particle confined to a one-dimensional finite interval $[0, 1]$. Whilst a point particle could simply remain at rest in this interval, and have zero kinetic energy, the wave-function $\psi(x)$ has an kinetic energy associated with (half) the integrated square of its derivative: $\int_0^1 \{\psi'(x)\}^2 \mathrm{d}x$. So far this might look something like the kinetic energy of a fluid, but there is a subtlety which makes a wave-function completely different from a classical fluid. The energy is actually determined by the *ratio*

$$\frac{\int_0^1 \{\psi'(x)\}^2 \mathrm{d}x}{\int_0^1 \{\psi(x)\}^2 \mathrm{d}x}$$

so that multiplying $\psi(x)$ by a constant makes no difference. The energy is a functional of the *shape* of the $\psi$, not of its scale. In particular, $\psi \equiv 0$ makes no sense in this ratio, so there is no obvious analogue of a particle at rest. Instead, the question of the *least value* taken by this ratio emerges, and it is far from intuitively clear. In fact, for functions such that $\psi(0) = 0 = \psi(1)$, the answer is $\pi^2$, as we shall show, and the existence of such a non-zero *ground-state energy* is a typical feature of quantum systems in much more general settings.

We need a formalism that will handle this problem, but also the more general problems that arise when the energy functional is not so simple, and the geometry of the space not just a simple interval. Clearly, the theory of stationary integrals, subject to an integral constraint, provides just this formalism. The ratio problem as described above can be restated as the problem of finding a minimum for $\int_0^1 \{\psi'(x)\}^2 \mathrm{d}x$ subject to the constraint that $\int_0^1 \{\psi(x)\}^2 \mathrm{d}x = 1$.

Thus the ratio of interest can be identified with the value taken by the Lagrange multiplier $\lambda$ in the solution. If we look at this in the light of the preceding discussion, we see that the relationship of $I[y]$ to $J[y]$ is very simple in this case: it is simply linear, $I = \lambda J$, with $\lambda$ interpretable as a constant price. But what is new in this situation is that for the first time we are taking seriously the fact that there are many local extrema, in fact a countable infinity of them, and we are studying how they inter-relate.

The inter-relation of the extrema is naturally expressed by seeing that $\lambda$ also takes on a further meaning as the *eigenvalue* of an differential operator.

8.2. **The Sturm-Liouville equation.** The differential operators we are concerned with are just the same as you have met in last term's course, but written

in a slightly different way. The standard *Sturm-Liouville form* is

(45)                    $(p(x)y')' + q(x)y = -\lambda r(x)y$    for  $a \le x \le b$,

with boundary conditions which will be specified below. Here $p, q, r$ are taken to have continuous derivatives, and we shall assume $r > 0, p \ge 0$.

It is immediate that this is the Euler-Lagrange equation for the variational problem of finding stationary values of

$$I[y] = \int_a^b (p(y')^2 - qy^2)\mathrm{d}x$$

subject to

$$J[y] = \int_a^b ry^2\mathrm{d}x = \text{ constant.}$$

We can now note the boundary conditions that are consistent with this interpretation: we have the usual choice between fixed and natural boundary conditions at each end, so that either

(46)                    $y(a) = 0$  or  $p(a)y'(a) = 0$,

and similarly at $b$.

8.3. **Examples.** 1.  If $p \equiv 1, q \equiv 0, r \equiv 1$, we regain the motivating example that began this section. But now we can solve it: the allowed values of $\lambda$ are just the sequence $\lambda_n = n^2\pi^2$, and the corresponding $y_n(x)$ are (proportional to) $\sin(n\pi x)$.

2.  If $p(x) = 1 - x^2, q \equiv 0, r \equiv 1$, we have *Legendre's equation* on $[-1, 1]$. With natural boundary conditions, the solutions are the Legendre polynomials $P_n(x)$, as met in the Differential Equations course.

3.  If $p(x) = x, q(x) = -k^2/x, r(x) = x$, we obtain the equation

$$(xy')' - \frac{k^2}{x}y = -\lambda xy$$

which is equivalent to

$$y'' + \frac{1}{x}y' - \frac{k^2}{x^2}y = -\lambda y$$

also recognisable from the Differential Equations course as *Bessel's equation* of order $k$. This has solutions vanishing at $x = 0$ of form $J_k(\lambda x)$. A fuller treatment would bring in the solutions to Bessel's equation which diverge at $x = 0$, but in the simplest situation, when boundary conditions $y = 0$ at $x = 0, x = a$, are imposed, there will be a discrete spectrum of $\lambda_n$ such that $J_k(\lambda_n x)$ satisfies them.

8.4. **Eigenfunction expansions.** The idea of Sturm-Liouville theory is to generalise the Fourier analysis that is naturally associated with case (1). From Mods, you know how to expand a general function in terms of sines and cosines, making use of their completeness and orthogonality properties. It turns out that these properties are not unique to the trigonometrical functions. They can be regarded as following from their emergence as solutions to a Sturm-Liouville ODE, and any other Sturm-Liouville equation will give rise to another set of functions with these completeness and orthogonality properties. That is, there is in general, for any Sturm-Liouville equation, a sequence of eigenfunctions $y_n(x)$ with completeness and orthogonality properties, such that a general function can usefully be expanded as $\sum_n c_n y_n$.

The proper statement and proof of this lies beyond this course (remember that even for Fourier theory the question of completeness is subtle, with great attention being needed for points of discontinuity.) However, we can show how the vital orthogonality properties emerge directly from this formulation.

Suppose, for a $(p, q, r)$ Sturm-Liouville system, we have two solutions $y_n, y_m$ with the corresponding $\lambda_n \neq \lambda_m$.

We will first verify that $\lambda_n$, the eigenvalue associated with the eigenfunction $y_n$, is equal to the quotient $I[y_n]/J[y_n]$ and so to the Lagrange multiplier in the integral formulation. We have

$$(p(x)y_n')' + q(x)y_n = -\lambda_n r(x)y_n$$

so multiplying by $y_n$ and integrating,

$$\int_a^b (p(x)y_n')'y_n + q(x)y_n^2 \, \mathrm{d}x = -\lambda_n \int_a^b r(x)y_n^2 \, \mathrm{d}x$$

so

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(py_n'y_n) \, \mathrm{d}x - \int_a^b (p(x)y_n'^2 - q(x)y_n^2) \, \mathrm{d}x = -\lambda_n \int_a^b r(x)y_n^2 \, \mathrm{d}x$$

i.e.

(47) $$[py_n'y_n]_a^b - I[y_n] = -\lambda_n J[y_n] \, .$$

But the boundary term vanishes because $a$ we either have $y_n(a) = 0$ or we have $p(a)y_n'(a) = 0$, and similarly at $b$. Hence $I[y_n] = \lambda_n J[y_n]$ as required.

Now we shall show that $y_m, y_n$ are orthogonal, in the sense that

(48) $$\int_a^b r y_n y_m \mathrm{d}x = 0 \, .$$

We have that

$$(p(x)y_n')' + q(x)y_n = -\lambda_n r(x)y_n$$
$$(p(x)y_m')' + q(x)y_m = -\lambda_m r(x)y_m$$

Multiplying the first by $y_m$, the second by $y_n$, subtracting, and integrating from $a$ to $b$,

$$\int_a^b (y_m(py_n')' - y_n(py_m')')\mathrm{d}x = -(\lambda_n - \lambda_m)\int_a^b r(x)y_m y_n \mathrm{d}x$$

But the LHS can be exactly integrated to

$$[p(y_m y_n' - y_n y_m')]_a^b$$

and thus vanishes by the boundary conditions, for at $a$ we either have $y_m(a) = 0 = y_n(a)$ or we have $p(a)y_m'(a) = 0 = p(a)y_n'(a)$, and similarly at $b$. Thus the RHS vanishes, but since $\lambda_m - \lambda_n \neq 0$ by assumption, the orthogonality follows.

This argument is the same as that used in the Algebra course where the general definition of an inner product is discussed. We have in effect defined an inner product structure on a space of functions by using the $r(x)$. As in Algebra, we can define an *orthonormal* set of basis functions $y_n$, with respect to this inner product, by choosing the scale such that

$$(49) \qquad J[y_n] = \int_a^b r(x)\{y_n(x)\}^2 \mathrm{d}x = 1\,.$$

8.5. **Rayleigh-Ritz approximation.** Throughout the course we have emphasised that the variational formalism is a two-way street. Our theory allows the solution, via differential equations, of notable problems involving extremals. On the other hand, it can be used profitably to reformulate problems to do with differential equations in terms of stationary integrals. In the context of Sturm-Liouville equations, the spectrum eigenvalues can be usefully investigated by calculating the integrals $I[y]$ and $J[y]$. In particular, trying out any $y$ whatever gives an upper bound for the lowest eigenvalue $\lambda_1$.

Thus, returning to the original example of

$$\frac{\int_0^1 \{\psi'(x)\}^2 \mathrm{d}x}{\int_0^1 \{\psi(x)\}^2 \mathrm{d}x}$$

we can try the simplest possible $y$ satisfying the boundary conditions, $y = x(1-x)$, and calculate

$$Q = \frac{\int_0^1 (2x-1)^2 \mathrm{d}x}{\int_0^1 x^2(1-x)^2 \mathrm{d}x} = 10$$

so that $\lambda_1 \leq 10$. This is a good approximation to $\lambda_1 = \pi^2$.

This process can be refined. Clearly, this approximation could be improved by optimising the trial $y$ over a set of parameters. Hence we arrive at a good approximation $\tilde{y}_1$ to $y_1$. Then, the next eigenvalue could be estimated by optimising over another class of trial functions, all orthogonal to $\tilde{y}_1$. This gives an estimate of $\lambda_2$ and $y_2$, and so on.

There is a reason why the approximation of eigenvalues is good; if the trial function $\tilde{y}_1$ is correct to $O(\epsilon)$, the eigenvalue $\lambda_1$ will be good to $O(\epsilon^2)$. For if

$$\tilde{y}_1 = y_1 + \sum_2^\infty c_n y_n$$

where each $c_n$ is of $O(\epsilon)$. then

$$I[\tilde{y}_1] = \lambda_1 + \sum_2^\infty \lambda_n |c_n|^2, \ J[\tilde{y}_1] = 1 + \sum_2^\infty |c_n|^2$$

whence $Q[\tilde{y}_1]$ differs from $\lambda_1$ by $O(\epsilon^2)$.

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