**Calculus of Variations - Problem Sheet 2**

**Trinity Term 2020**

1. It is required to find an extremal of the functional

$$\int_a^b F(x, y(x), y'(x), y''(x))dx$$

among all smooth functions $y(x)$ which satisfy the boundary conditions

$$y(a) = y(b) = 0.$$

Show that such an extremal must be a solution of the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

and must satisfy the natural boundary conditions

$$\frac{\partial F}{\partial y''} = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = b.$$

2. An elastic beam has vertical displacement $y(x), x\epsilon[0, l]$. (The $x$-axis is horizontal and the $y$-axis is vertical and directed upwards.) The ends of the beam are supported, that is, $y(0) = y(l) = 0$, and the displacement minimizes the energy

$$\int_0^l \{\frac{1}{2}D[y''(x)]^2 + \rho g y(x)\}dx,$$

where $D, \rho$ and $g$ are positive constants. Write down the differential equation and the boundary conditions that $y(x)$ must satisfy and show that

$$y(x) = -\frac{\rho g}{24D}x(l - x)(l^2 + x(l - x)).$$

3. Find an extremal corresponding to

$$\int_{-1}^1 y dx$$

when subject to $y(-1) = y(1) = 0$ and

$$\int_{-1}^1 (y^2 + y'^2)dx = 1$$

4 (a) Suppose that $F : \mathbb{R}^7 \to \mathbb{R}$ is a $C^2$-function and that the $C^2$-function $u : \mathbb{R}^3 \to \mathbb{R}$ gives a stationary value to the integral

$$\int\int\int_{\vee} F(x, y, z, u, u_x, u_y, u_z)dxdydz,$$

and satisfies $u = f$ on the smooth simple closed surface $\partial\vee$ which bounds the open set $\vee$ in $\mathbb{R}^3$. Show that $u$ satisfies the Euler equation

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = \frac{\partial F}{\partial u}$$

(b) Let $\vee = \{(x, y, z)\epsilon\mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. Find an extremal $u = u(x, y, z)$ for the problem of minimizing the integral

$$\int\int\int_{\vee}(u_x^2 + u_y^2 + u_z^2)dxdydz$$

when subject to the constraints

$$\int\int\int_{\vee} u \quad dxdydz = 4\pi$$

and $u = 1$ on the boundary of $\vee$

5. Let $p$ be a positive real-valued function differentiable on the bounded interval $[a, b]$ and let $q$ and $r$ be positive real-valued continuous functions on $[a, b]$. Show that the extremals of

$$J(y) = \int_a^b \left(py'^2 + qy^2\right) dx$$

subject to the constraint

$$\int_a^b ry^2 dx = 1$$

must satisfy

$$(py')' + (-q + \lambda r)y = 0 \qquad (A)$$

with $py' = 0$ at $x = a$ and $x = b$.

Show that if $y_1$ and $y_2$ are solutions to (A) for $\lambda = \lambda_1, \lambda_2$ respectively, where $\lambda_1 \neq \lambda_2$, then

$$\int_a^b ry_1y_2dx = 0. \qquad (B)$$

Find the extremals of $\displaystyle\int_0^{\pi} y'^2dx$ subject to $\displaystyle\int_0^{\pi} y^2dx = 1$ and the corresponding values of $\lambda$.Verify that these extremals satisfy (B).