INTRODUCTION TO MANIFOLDS

(Multivariable calculus)

analysis

geometry

Q: What does it mean for a function $f: \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable?

$x_1, \dots, x_n$

$$f(x_1, x_2, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

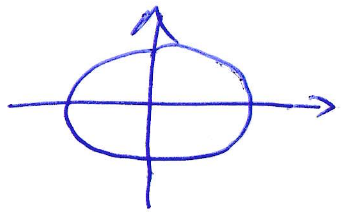
Q1 $n = m$ invertibility of f ?

Q2 $n \geq m$ $f(x, y) = 0$ with $x \in \mathbb{R}^{n-m}$, $y \in \mathbb{R}^m$

When is there a "nice" solution $y = g(x)$

Ex $n = 2$, $m = 1$, $f(x, y) = x^2 + y^2 - 1$

$f(x, y) = 0$



$y = \pm \sqrt{1 - x^2}$

Q3 any n, m $f^{-1}(0) \subset \mathbb{R}^n$ - when is it "nice" ?

The answers to all these questions will turn out to be local

- f may not be defined on the whole of $\mathbb{R}^n$
- the answer might depend on where we are - "local on $\mathbb{R}^n$"

$$\boxed{f : \Omega \longrightarrow \mathbb{R}^m}$$

$$f : (\Omega \subset \mathbb{R}^n) \to \mathbb{R}^m$$

$\Omega \subset \mathbb{R}^n$, open (contains an open ball around any of its points)
and connected ($\Leftrightarrow$ path connected)

It will be enough to assume that Ω is a ball (open)

$n = m = 1$ $\quad \Omega = I$ open interval in $\mathbb{R}$

$f : I \to \mathbb{R}$

is differentiable at $x \in I$ if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists}$$

Equivalently f is differentiable at x if there exists a number α_x such that

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x) - a_x \cdot h}{h} \right) = 0$$

Equivalently still f is differentiable at x if there exists a linear map $L_x : \mathbb{R} \to \mathbb{R}$, $h \mapsto \alpha_x \cdot h$ such that

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x) - L_x(h)}{h} = 0}$$

$n = 1$, m arbitrary $f(x) = (f_1(x), \dots, f_m(x))$ defined on $I \subset \mathbb{R}$.

f is differentiable at x if and only if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

This is equivalent to...

... This is equivalent to:

each $f_i'(x)$ existing, <u>and</u> $f'(x) = (f_1'(x), \ldots, f_n'(x))$

x, h: numbers, $f'(x)$ vector in $\mathbb{R}^n$

This is equivalent to: there exists a linear map $L_x : \mathbb{R} \to \mathbb{R}^n$

$h \mapsto (f_1'(x) \cdot h, \ldots, f_n'(x) \cdot h)$

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L_x(h)}{h} = 0$$

Slogan: "L_x is a good linear approximation to f near x"

Question: how to extend this to n, m arbitrary?

Challenge: cannot divide by vectors, but: can measure vector size/length!

Comment $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$f(x_i) = (f_1(x_i), \ldots, f_m(x_i))$$

Can of course form $\left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$

Want a coordinate-independent definition of "derivative"

But also when taking coordinates, we want to compute

$$\rightsquigarrow \frac{\partial f_i}{\partial x_j}.$$

Tools We are going to use Euclidean length / distance on $\mathbb{R}^n$, $\mathbb{R}^m$

For $x \in \mathbb{R}^n$, $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ is a distance function/ metric

in the sense that

- $\|x\| = 0 \iff x = 0$
- $\|x\| + \|y\| \geq \|x+y\|$
- $\|x\| \geq 0$

Also $\|\lambda x\| = |\lambda| \, \|x\|$

Comment Looks like coordinate dependent - does not matter...

Extension: we can also use this for matrices

$C \in Mat_{n \times m}(\mathbb{R})$

has Hilbert - Schmidt norm

$$\|C\| = \left(\sum_{i,j=1}^{n} c_{ij}^2 \right)^{1/2}$$

and then we have the useful inequality that for $h \in \mathbb{R}^n$,

$$\|C \cdot h\| \leq \|C\| \cdot \|h\| \qquad C \in Mat_{n \times n}(\mathbb{R}),\ h \in \mathbb{R}^n$$

also

$$\|A \cdot B\| \leq \|A\| \cdot \|B\| \qquad A, B \in Mat_{n \times n}(\mathbb{R})$$

__Examples__

i.) $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$x \longmapsto |x|$

ii.) $f : \mathbb{R}^3 \setminus \{p\} \longrightarrow \mathbb{R}^3$

$x \longmapsto q \dfrac{x-p}{|x-p|^3}$ electric field of strength q at p

iii.) Coming from $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \exp(z)$ is the function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (e^x \cos y, \; e^x \sin y)$$

iv.) Functions defined on matrices, such as ...

$$f_i : \mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\mathrm{Mat}_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$$

$$f_1 : A \longmapsto \operatorname{tr} A$$

$$f_2 : A \longmapsto \det A$$

or, fixing B, can define

$$f_B : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$$

$$A \longmapsto A \cdot B$$

Finally

N., $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \left(x^2 + \cos y - y/_{x^2+1},\ x^2 + y^3 - 3\cos y,\ \exp(xy)\right)$$

Partial derivatives

$f: \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, consider $e_1, \dots, e_n$ standard basis vectors of $\mathbb{R}^n$

Then the i^{th} partial derivative of f exists at $x \in \Omega$ if

$$\lim_{t \to 0} \frac{f(x+te_i) - f(x)}{t} \quad \text{exists, denoted} \quad \frac{\partial f}{\partial x_i}(x) \in \mathbb{R}^m$$

Also denoted $\partial_i f(x)$, $D_i f(x)$ ...

Examples i) $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$x \longmapsto |x|$

We want to investigate

$$\lim_{t \to 0} \frac{|x+te_i| - |x|}{t} \quad ?$$

$$\frac{1}{t} \frac{|x+te_i|^2 - |x|^2}{|x+te_i| + |x|} = \frac{1}{t} \frac{2tx_i + t^2}{|x| + |x+te_i|} = \frac{2x_i + t}{|x| + |x+te_i|}$$

$$\xrightarrow[\text{as } t \to 0]{} \frac{x_i}{|x|} \quad \text{as long as } x \neq 0.$$

So $\boxed{\partial_i f(x) = \frac{x_i}{|x|} \text{ if } x \neq 0, \text{ no limit if } x = 0.}$

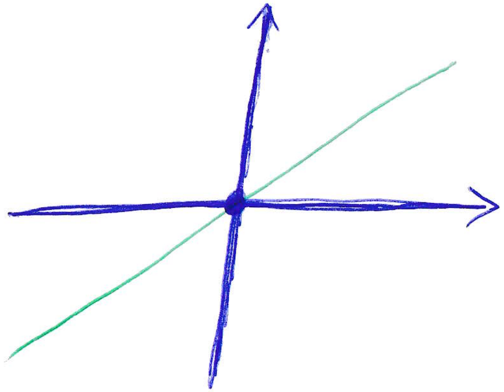
ii.) (Key example) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \begin{cases} xy/(x^2+y^2)^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = 0 \end{cases} \qquad (*)$$

a.) At a point $(x,y) \neq (0,0)$ and in a small ball around (x,y), have a nice formula $(*)$ for our function. Here we can just differentiate "using usual rules"

$$\left.\begin{aligned} \partial_x f(x,y) &= \frac{y}{(x^2+y^2)^2} - \frac{4x^2y}{(x^2+y^2)^3} \\ \partial_y f(x,y) &= \dots \end{aligned}\right\} \text{exist}$$

b) At $(0,0)$:



Function f is zero along both axes!

So $\partial_x f(0,0) = 0$ (limit exists! identically)

$\partial_y f(0,0) = 0$ (- - -)

<u>But</u> f is <u>not even continuous</u> at $(0,0)$

To see this, note that on the line $x = y$

$$f(x,x) = \frac{1}{4x^2}$$ blows up at the origin!

This function has all partial derivatives but is not even continuous.