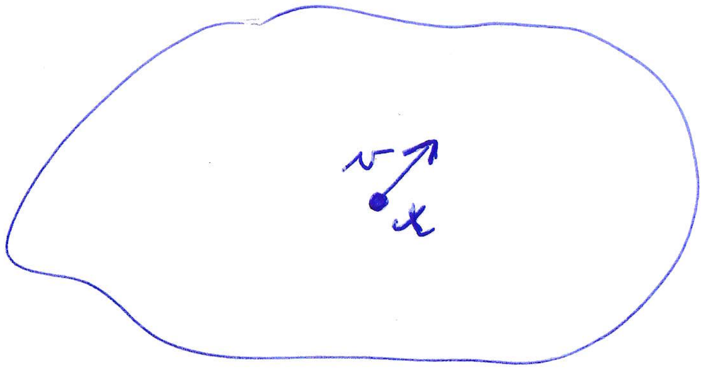
Definition Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, and $v \in \mathbb{R}^n \setminus \{0\}$.

The directional derivative of f at $x \in \mathbb{R}^n$ in direction v is

$$\partial_v f(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

if the limit exists.



$$\Omega \xrightarrow{f} \mathbb{R}^m$$

Note that with $v = e_i$, we recover $\partial_i f(x) = \partial_{e_i} f(x)$.

Ex $\quad f(x) = |x| \qquad f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\partial_v f(x) = \left.\frac{d}{dt}\right|_{t=0} |x + tv| = \left\langle \frac{x}{|x|}, v \right\rangle \text{ if } x \neq 0.$$

Calculation in printed lecture notes

Remark: think about what this says for $n = 1$.

__Definition__ $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, assume all partials exist at $x \in \Omega$, then

$$\nabla f(x) = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{pmatrix}$$

__Note__ that we have

$$\partial_v f(x) = \left\langle \nabla f(x), v \right\rangle$$

and so, for example, for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto |x|$, $\nabla f(x) = \dfrac{x}{|x|}$.

Definition For general $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that all partial derivatives $\partial_i f_j$ at x exist for $x \in \Omega$, then define the Jacobian matrix

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \partial_2 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & & & \vdots \\ \partial_1 f_m(x) & \cdots & \cdots & \partial_n f_m(x) \end{pmatrix}$$

If $n = m$, then we have the Jacobian determinant

$$J_f(x) = \det(Df(x))$$

Function spaces $C^k(\Omega, \mathbb{R}^m)$

Given a domain $\Omega \subset \mathbb{R}^n$.

We are looking at functions $f : \Omega \longrightarrow \mathbb{R}^n$

k^{th} partial derivatives for an index set $i_1, \dots, i_k \in \{1, \dots, n\}$: let

$x \in \Omega$, $\partial_{i_1, i_2, \dots, i_k} f := \partial_{i_1} (\partial_{i_2} (\partial_{i_3} \dots \partial_{i_k} f) \dots) (x)$

if all relevant limits exist.

$f \in C^k(\Omega, \mathbb{R}^n)$ if for all $x \in \Omega$, and all possible index sets, k^{th} derivatives exist and are continuous at x.

Theorem of Schwarz If $f \in C^2(\Omega, \mathbb{R}^n)$, then $\forall\, 1 \le i,j \le n$ and $x \in \Omega$,

$$\boxed{\partial_{i,j} f(x) = \partial_{j,i} f(x)}$$

i.e. order of differentiation can be exchanged.

Corollary If $f \in C^k(\Omega, \mathbb{R}^n)$ then for every index set $i_1, \dots, i_k$, the order of the partial differentiations can be taken arbitrarily.

Proof of Corollary: interchange orders one step at a time, using Thm of Schwarz. 2-cycles generate all permutations.

Example / Exercise

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$$

Check

a.) $f \in C^1(\mathbb{R}^2)$ [for $n=1$, write $C^k(\Omega) = C^k(\Omega, \mathbb{R})$]

b.) $\partial_x \partial_y f(0,0) = 1$

$\partial_y \partial_x f(0,0) = -1$

So existence of the limits defining $\partial_x \partial_y$, $\partial_y \partial_x$ is not sufficient for their equality.

Definition *** $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $x \in \Omega$ if there exists a linear map $L_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L_x(h)}{|h|} = 0$$

We say f is differentiable on Ω if it is so $\forall x \in \Omega$.

Alternatively, we require $L_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$f(x+h) = f(x) + L_x(h) + R_f(h)$$

Constant term at x

linear term

remainder term "small compared to h"

such that $\boxed{\lim_{h \to 0} \frac{R_f(h)}{|h|} = 0.}$

Note: for $n = m = 1$, we are back to the interpretation of $f'(x)$ as the slope of the tangent line!

Linear and quadratic cases

Let A be an $n \times m$ matrix, $f : \mathbb{R}^n \to \mathbb{R}^m$
$b \in \mathbb{R}^m$ $\qquad x \longmapsto Ax + b$

$$f(x+h) - f(x) = Ah$$

So can take $L_x(h) = Ah$, $R_f(h) = 0$.

In other words, $L_x = A$ $\qquad$ Linear maps stay themselves as derivatives

Quadratic case: <u>quadratic forms</u>. Take $C \in \mathrm{Sym}_n(\mathbb{R})$

and let $f(x) = x^t C x$, this defines $f: \mathbb{R}^n \longrightarrow \mathbb{R}$.
$\quad = \langle x, Cx \rangle$

$$f(x+h) - f(x) = \langle x+h, C(x+h) \rangle - \langle x, Cx \rangle$$

$$= \langle x, Ch \rangle + \langle h, Cx \rangle + \langle h, Ch \rangle$$

$\langle , \rangle$ symm. $\longrightarrow$ $= \langle Ch, x \rangle + \langle h, Cx \rangle + \langle h, Ch \rangle$

$C = C^t$ $\longrightarrow$ $= \underbrace{2\langle h, Cx \rangle}_{\text{candidate for } L_x(h)} + \langle h, Ch \rangle$

Indeed, this is correct, since ...

$$\frac{|f(x+h) - f(x) - L_x(h)|}{|h|} = \frac{|\langle h, Ch\rangle|}{|h|} \leq$$

$$\leq \frac{|h|\,|Ch|}{|h|} \leq \|C\|\,|h|$$

So indeed, $\frac{R_f(h)}{|h|} \to 0$ as $h \to 0$.

Hence indeed, f is differentiable at every $x \in \mathbb{R}^n$ and

$$\boxed{L_x(h) = 2\langle h, Cx\rangle}$$

Last example

$$\begin{tikzcd} f : \mathbb{R}^{n^2} \arrow[r] \arrow[d, equal] & \mathbb{R}^{n^2} \arrow[d, equal] \\ \mathrm{Mat}_{n\times n}(\mathbb{R}) & \mathrm{Mat}_{n\times n}(\mathbb{R}) \end{tikzcd}$$

$$A \longmapsto A^2$$

For $H \in \mathbb{R}^{n^2}$ (small)

$$\begin{aligned} f(A+H) - f(A) &= (A+H)^2 - A^2 \\ &= \underbrace{AH + HA}_{\text{linear}} + H^2 \end{aligned}$$

constant

remainder "higher order"

So, with $\boxed{L_A(H) = AH + HA}$, get

$$\frac{\| f(A+H) - f(A) - L_A(H) \|}{\| H \|} = \frac{\| H^2 \|}{\| H \|} \leq \frac{\| H \|^2}{\| H \|} \longrightarrow 0$$

as $H \longrightarrow 0$.

Denote $\quad df(x) = L_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (if exists)

Theorem A If $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $x \in \Omega$ then f is continuous, all partial derivatives exist at x and

$$\underbrace{df(x)}_{\substack{\text{linear map} \\ \mathbb{R}^n \to \mathbb{R}^m}}(h) = \underbrace{Df(x)}_{\text{matrix}} \cdot h$$

matrix multiplication

Theorem B If $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has continuous partial derivatives at $x \in \Omega$, then f is differentiable at $x \in \Omega$.