Recall $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is called differentiable at $x \in \Omega$ if there exists a linear map $df(x) = L : \mathbb{R}^n \to \mathbb{R}^m$ such that near x we have

$$f(x+h) = f(x) + L(h) + R_f(h)$$

with $\lim\limits_{h \to 0} \dfrac{R_f(h)}{|h|} = 0.$ $\quad (*)$

Note/define $h \to 0$ means $\begin{cases} |h| \to 0 & \text{or, equivalently} \\ \forall i,\ h_i \to 0, & \text{here } h = \sum_{i=1}^{n} h_i e_i \end{cases}$

Note: $|h| = \left(\sum_{i=1}^{n} h_i^2 \right)^{1/2}$

Typically, to check $(*)$, what we really check is

$$\frac{|R_f(h)|}{|h|} \longrightarrow 0 \quad \text{as } |h| \to 0$$

Theorem A If $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \Omega$ then f is continuous; also the partial derivatives exist, and

$$\underbrace{df(x)}_{\text{linear map } \mathbb{R}^n \to \mathbb{R}^m}(h) = \underbrace{Df(x)}_{\text{its matrix}} \cdot h$$

Proof

$$\lim_{h \to 0} \left(f(x+h) - f(x) \right) = \lim_{h \to 0} \left(\underbrace{L(h)}_{\substack{\downarrow \text{ as } |h| \to 0 \\ 0}} + \underbrace{R_f(h)}_{\substack{\downarrow \text{ as } |h| \to 0 \\ 0}} \right)$$

since $|L(h)| \leq \|M\| \cdot |h|$ for the matrix M of L.

$= 0$ which implies continuity.

To show that $\partial_i f(x)$ exists, put $h = te_i$, then differentiability condition becomes

$$\lim_{t\to 0}\Big(\underbrace{\frac{1}{t}\big(f(x+te_i)-f(x)\big)}_{\text{if has limit at } t\to 0, \text{ it is } \partial_i f(x)} - L(e_i)\Big) = 0$$

So we get $\quad \partial_i f(x) = L(e_i) \in \mathbb{R}^m$

Finally $L(h) = L\left(\sum_{i=1}^{n} h_i e_i\right) = \sum_{i=1}^{n} h_i L(e_i) =$

$$= \sum_{i=1}^{n} h_i \underbrace{\partial_i f(x)}_{\text{gives } Df(x)}$$

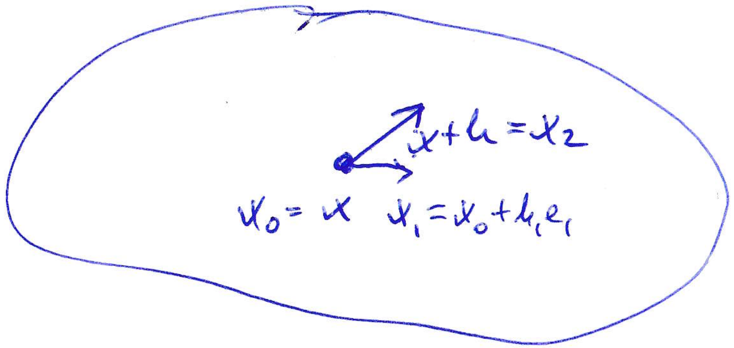
This indeed means $L(h) = Df(x_i)\cdot h.$ $\square$

__Theorem B__ Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ has __continuous__ first partial derivatives at $x \in \Omega$. Then f is differentiable at $x \in \Omega$.

__Proof__ We can assume $m = 1$ (Otherwise, work componentwise, and combine using Euclidean length formula).
We are at $x \in \Omega \subset \mathbb{R}^n$; $m = 1$.

$$\Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$

(draw $n=2$)



Candidate linear map $L(h) = \sum_{i=1}^{n} \partial_i f(x) \cdot h_i$

We need to estimate $f(x+h) - f(x) - L(h)$

Let $x_0 = x$, $x_1 = x + h_1 e_1$, $x_2 = x + h_1 e_1 + h_2 e_2$, $\dots$ $x_n = x + h$

where $h = \sum_{i=1}^{n} h_i e_i$

<u>Key point</u>: along coordinate axes, f can be treated as a single-variable function.

$$f(x+h) - f(x) = f(x_n) - f(x_0)$$
$$= \sum_{i=0}^{n-1} \left(f(x_{i+1}) - f(x_i) \right)$$

Now $f(x_k) - f(x_{k-1}) = f(x_{k-1} + \underline{\underline{h_k}}\, e_k) - f(x_{k-1})$

only variable that changes

So $f(x_k) - f(x_{k-1}) = \partial_k f(x_{k-1} + \theta_k h_k e_k) \cdot h_k$ (+)

for some $\theta \in [0,1]$ by the single-variable Mean Value Theorem.

Now

$$\frac{|f(x+h)-f(x)-L(h)|}{|h|} = \frac{1}{|h|}\left|\sum_{k=1}^{n} f(x_k)-f(x_{k-1}) - \sum_{k=1}^{n} \partial_k f(x)\, h_k\right|$$

$$\overset{(+)}{=} \frac{1}{|h|}\left|\sum_{k=1}^{n} \big((\partial_k f)(x_{k-1}+\theta_k h_k e_k) - \partial_k f(x)\big) h_k\right|$$

Cauchy-Schwartz $\longrightarrow$ $$\leq \frac{1}{\cancel{|h|}} \underbrace{\left(\sum_{k=1}^{n} \big(\partial_k f(x_{k-1}+\theta_k h_k e_k) - \partial_k f(x)\big)^2\right)^{1/2}}_{\downarrow \text{ as } h \to 0 \atop 0} \cdot \cancel{|h|}$$

by continuity of $\partial_k f$ at $x \in \Omega$.

$\square$

Corollary

$$\begin{tikzcd} \forall \varepsilon\ \partial_\varepsilon f \text{ continuous} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & f \text{ continuous.} \\ & f \text{ diff'able} \arrow[ur, Rightarrow] & \end{tikzcd}$$

Proof

Note $f \in C^1(\Omega, \mathbb{R}^m) \Longrightarrow f$ differentiable.

Remember Existence of $\partial_\varepsilon f(x)$ for all ε does not implies any of these!!!

Proposition (The Chain Rule)

Let $\Omega \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be domains. Let $f : V \to \mathbb{R}^k$, $g : \Omega \to V$

Suppose g is differentiable at $x \in \Omega$, f is differentiable at $y = g(x) \in V$. Then

$$f \circ g : \Omega \longrightarrow \mathbb{R}^k$$

is differentiable at $x \in \Omega$; and

$$\boxed{d(f \circ g)(x) = df(g(x)) \circ dg(x)}$$

composition of linear maps

$$\mathbb{R}^n \xrightarrow[dg(x)]{} \mathbb{R}^m \xrightarrow[df(g(x))]{} \mathbb{R}^k$$

Proof $\quad A = dg(x), \quad B = df(g(x)) = df(y)$

$$g(x+h) = g(x) + Ah + R_g(h) \quad , \quad h \in \mathbb{R}^n, \quad \frac{|R_g(h)|}{|h|} \to 0$$

as $h \to 0$

$$(*) \quad f(y+\eta) = f(y) + B\eta + R_f(\eta) \quad , \quad \eta \in \mathbb{R}^m, \quad \frac{|R_f(\eta)|}{|\eta|} \to 0$$

as $\eta \to 0$

Choose $\eta = g(x+h) - \underbrace{g(x)}_{y} = Ah + R_g(h)$; note $h \to 0$ will imply $\eta \to 0$.

$y + \eta = g(x+h)$; $(*)$ becomes

$$f(g(x+h)) = f(g(x)) + B\left(Ah + R_g(h)\right) + R_f(Ah + R_g(h))$$

$$= \underbrace{f(g(x))}_{\text{Constant term}} + \underbrace{BA.h}_{\text{linear}} + \underbrace{BR_g(h) + R_f(Ah + R_g(h))}_{\text{remainder}}$$

<u>Claim</u> $$\lim_{h \to 0} \frac{|BR_g(h) + R_f(Ah + R_g(h))|}{|h|} = 0$$

<u>Proof of Claim</u> $$\frac{|BR_g(h)|}{|h|} \leq \|B\| \cdot \frac{|R_g(h)|}{|h|} \longrightarrow 0 \text{ as } |h| \to 0$$

Also, for h small enough,

$$|Ah + R_g(h)| \leq \|A\| \cdot |h| + |R_g(h)| \leq (\|A\| + 1)|h|$$

for $|h|$ sufficiently small. Hence

$$\frac{|R_f(Ah + R_g(h))|}{|h|} \longrightarrow 0 \quad \text{as } |h| \to 0$$

from properties of R_f.

$\square$

Corollary Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is invertible, with inverse $g: \mathbb{R}^n \to \mathbb{R}^n$ (or on some domains...). Suppose f is diff'able at $x \in \mathbb{R}^n$ and g is differentiable at $f(x) \in \mathbb{R}^n$. Then

$$dg(f(x)) = (df(x))^{-1}$$

as linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

Proof Chain Rule.

Note This implies in particular that $df(x)$ is invertible, so

$J_f(x) \neq 0$.