Introduction to Manifolds

Lecture 4

Balázs Szendrői, University of Oxford, Trinity term 2020

Recap from the end of last lecture

Proposition (Chain Rule) Let $\Omega \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open and connected sets, let $g\colon \Omega \to V$ and $f\colon V \to \mathbb{R}^k$. Suppose that g is differentiable at $x \in \Omega$ and f is differentiable at $y = g(x) \in V$. Then the map

$$f \circ g\colon \Omega \to \mathbb{R}^k$$

is differentiable at x and

$$d(f \circ g)(x) = df(g(x))dg(x)\,.$$

Corollary (Derivative of the Inverse) Let $\Omega \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ be open and connected sets, and suppose $f\colon V \to \Omega$ is invertible with inverse $g\colon \Omega \to V$. Suppose further that f is differentiable at $x \in V$ and that g is differentiable at $g = f(x) \in \Omega$.
Then

$$dg(f(x)) = (df(x))^{-1}.$$

The rule for the derivative and its invese in action

Example (Coordinate change)
Let $f\colon \mathbb{R}_+ \times (0, 2\pi) \subset \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$(x, y) = f(r, \varphi) = (r\cos\varphi, r\sin\varphi).$$

Let g be the inverse function to f. Then

$$Df(r, \varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

so we compute

$$\det Df(r, \varphi) = r > 0$$

and indeed $Df(r, \varphi)$ is invertible. Also

$$Dg(x, y) = Df(r, \varphi)^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

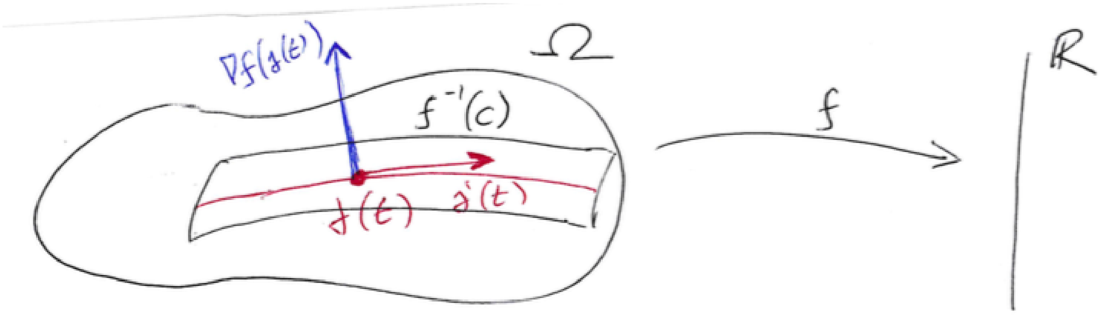
Gradient and level set

Corollary (The gradient is perpendicular to level sets) Let

$$f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$$

be differentiable and let $\gamma\colon (\alpha, \beta) \subset \mathbb{R} \to \Omega$ be a differentiable curve segment. Assume that the curve lies in a level set of f, that is $f(\gamma(t)) = c$ for all $t \in (\alpha, \beta)$. Then for all $t \in (\alpha, \beta)$, we have

$$0 = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle .$$



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$$0 = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle .$$

Proof By the constancy of $f(\gamma(t))$ and the Chain Rule, we have

$$\begin{aligned} 0 &= d(f \circ \gamma)(t) \\ &= df(\gamma(t))\gamma'(t) \\ &= \langle \nabla f(\gamma(t)), \gamma'(t) \rangle . \end{aligned}$$

A version of the Mean Value Theorem

Classical Mean Value Theorem for a continuous function $f : \mathbb{R} \to \mathbb{R}$ differentiable on the interval (x, y): for some $\xi \in (x, y)$,

$$f(x) - f(y) = f'(\xi)(x - y).$$

This does not readily generalise to the vector-valued context, since in general we get a different ξ for every component.

Proposition Suppose that

$$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$$

is differentiable. Let $x, y \in \Omega$ be such that the line segment

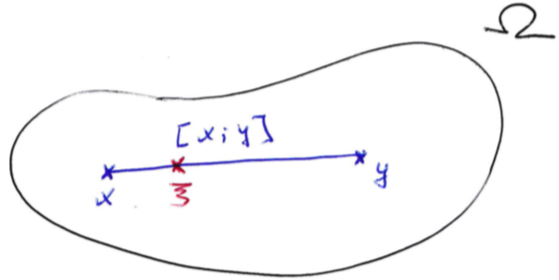
$$[x; y] = \{tx + (1 - t)y \,|\, t \in [0, 1]\}$$

is also contained in Ω.
Then there exists $\xi \in [x; y]$ such that

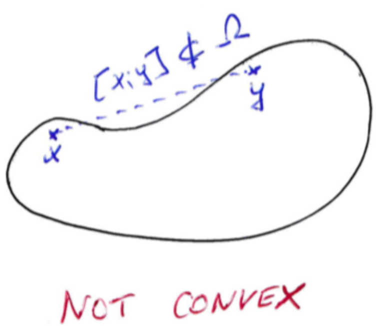
$$f(x) - f(y) = df(\xi)(x - y) = \langle \nabla f(\xi), x - y \rangle .$$

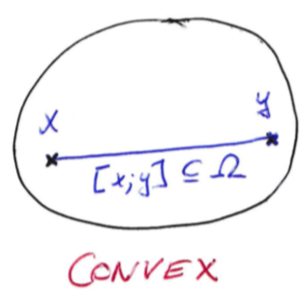
A version of the Mean Value Theorem: illustration

For $x, y \in \Omega$, the line segment $[x; y]$ is also contained in Ω:



Remark: if $[x; y] \subset \Omega$ is true for all pairs of points $x, y \in \Omega \subset \mathbb{R}^n$, then Ω is called **convex**:





A version of the Mean Value Theorem: proof

Proposition Suppose that $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable. Let $x, y \in \Omega$ be such that the line segment $[x; y]$ is also contained in Ω. Then there exists $\xi \in [x; y]$ such that $f(x) - f(y) = df(\xi)(x - y) = \langle \nabla f(\xi), x - y \rangle$.

Proof Let $\gamma(t) = tx + (1 - t)y, t \in [0, 1]$, and $F(t) = f(\gamma(t))$.
Then $f(x) = F(1)$ and $f(y) = F(0)$.
The Chain Rule implies that F is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t) = df(\gamma(t))\gamma'(t)\,.$$

By the classical Mean Value Theorem, there exists $\tau \in (0, 1)$ such that

$$F(1) - F(0) = F'(\tau).$$

Hence finally, with $\xi = \gamma(\tau)$,

$$f(x) - f(y) = df(\gamma(\tau))(x - y) = df(\xi)(x - y).$$

The Inverse and Implicit Function Theorems: Introduction

The Inverse Function Theorem and the Implicit Function Theorem are two of the most important theorems in multivariable analysis.

- The Inverse Function Theorem tells us when we can locally invert a function:

$$y = f(x) \overset{?}{\Longrightarrow} x = g(y)$$

- The Implicit Function Theorem tells us when a set of variables is given implicitly as a function of other variables.

$$f(x, y) = 0 \overset{?}{\Longrightarrow} y = g(x)$$

The flavour of these results is similar:

- We linearise the problem at a point p by considering the derivative $df(p)$.
- If a certain nondegeneracy condition on $df(p)$ holds, we obtain a result that works on a neighbourhood of the point p.

An example

We start with a simple example. Consider

$$S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

the unit circle in the plane. We can write

$$S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\}$$

for

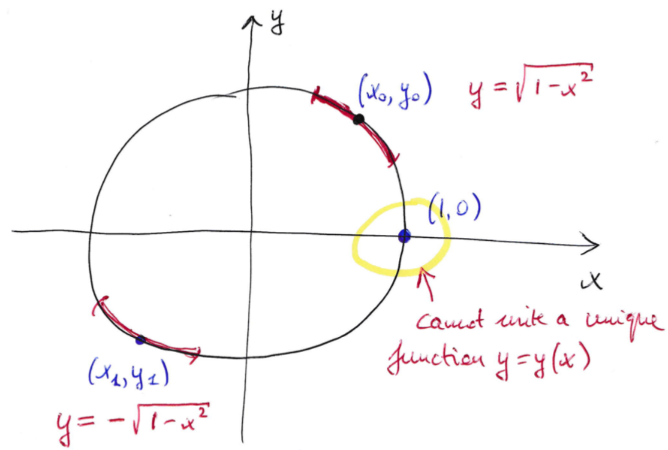
$$f(x, y) = x^2 + y^2 - 1.$$

Can we find a function $y = y(x)$ such that $x^2 + y(x)^2 = 1$?
Well, we could naively write

$$y(x) = \sqrt{1 - x^2}.$$

But there are issues: choice of sign; also issues in a neighbourhood of "bad" points.

An example (continuted)



The conclusion is that we can find $y = y(x)$ **locally**, in a neighbourhood of a point $(x_0, y_0) \in S^1$, **as long as** $y_0 \neq 0$.

We can write explicitly $y(x) = \sqrt{1 - x^2}$ if $y_0 > 0$ and $y(x) = -\sqrt{1 - x^2}$ if $y_0 < 0$, both for $|x| < 1$.

If $y_0 = 0$, we cannot find such a function $y = y(x)$.

The Implicit Function Theorem in $\mathbb{R}^2$

Theorem (Implicit Function Theorem in $\mathbb{R}^2$) Let $\Omega \subseteq \mathbb{R}^2$ be open and $f \in C^1(\Omega) = C^1(\Omega, \mathbb{R})$. Let $(x_0, y_0) \in \Omega$ and assume that

$$f(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x_0, y_0) \neq 0\,.$$

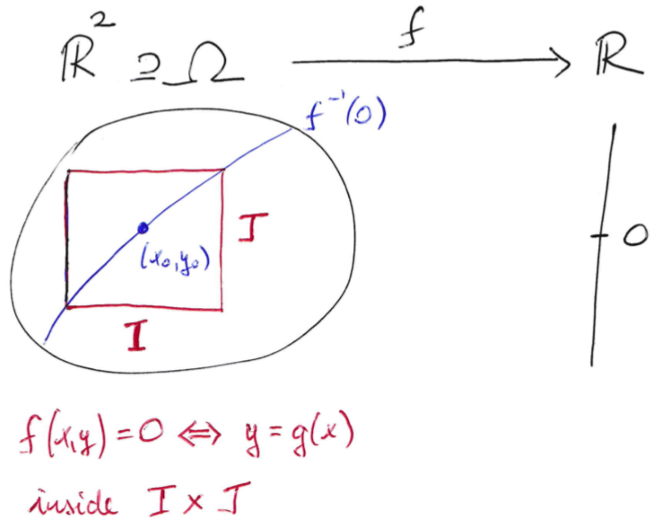
Then there exist open intervals $I, J \subseteq \mathbb{R}$ with $x_0 \in I$, $y_0 \in J$ and a unique function $g\colon I \to J$ such that $y_0 = g(x_0)$ and

$$f(x, y) = 0 \qquad \text{if and only if} \qquad y = g(x) \quad \text{for all } (x, y) \in I \times J.$$

Furthermore, $g \in C^1(I)$ with

$$g'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)}\,.$$

The Implicit Function Theorem in $\mathbb{R}^2$: illustration



I will not prove this result here. The proof in this case is easier than the general case, and can be found in the full set of Lecture Notes (non-examinable).

The Implicit Function Theorem in $\mathbb{R}^2$: return to the example

Consider again the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

for

$$f(x, y) = x^2 + y^2 - 1.$$

At a point $(x_0, y_0) \in S^1$, we have

$$\frac{\partial f}{\partial y}(x_0, y_0) = 2y_0.$$

So the condition $y_0 \neq 0$ that we found "by hand" for the existence of $y = y(x)$ precisely matches the condition of the Implicit Function Theorem.

The Implicit Function Theorem in $\mathbb{R}^2$: return to the example

Also, the function $y(x) = \sqrt{1-x^2}$ is differentiable away from $x = \pm 1$:

$$y'(x) = \frac{d}{dx}\sqrt{1-x^2} = \frac{-2x}{\sqrt{1-x^2}}$$

and for example at $(x_0, y_0) = (0, 1)$ we have

$$y'(0) = -\frac{\frac{\partial f}{\partial x}(0,1)}{\frac{\partial f}{\partial y}(0,1)} = -\frac{0}{2} = 0$$

so has horizontal tangent at $x = 0$ (check on the picture!).

Finally, note that while at $(\pm 1, 0)$ an expression $y = y(x)$ does not exist, we can switch the roles of x and y. In a neighbourhood of these points, we can write

$$x = \sqrt{1-y^2}$$

in agreement with the Implicit Function Theorem, since

$$\frac{\partial f}{\partial x}(\pm 1, 0) \neq 0.$$

Another example

Consider

$$C = \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\} \subset \mathbb{R}^2$$

for

$$f(x, y) = x^5 + x^2y^2 - 5x + 2y + y^5.$$

At the point $(0, 0) \in C$, we have

$$\frac{\partial f}{\partial y}(0, 0) = 2 \neq 0.$$

So by the Implicit Function Theorem, there exists a C^1 function $g : I \to J$ for small intervals I, J around 0 which can be used explicitly parametrise C by $y = g(x)$ in a neighbourhood of $(0, 0)$. This is however not going to be a function given by an explicit formula: all we can deduce is the existence of the function, its differentiability, as well as the value of its derivative at 0.

$$g'(0) = -\frac{\partial f}{\partial x}(0, 0) \Big/ \frac{\partial f}{\partial y}(0, 0) = 5/2.$$

Conclusion

For an arbitrary

$$C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\} \subset \mathbb{R}^2$$

with $f \in C^1(\Omega)$ for $\Omega \subset \mathbb{R}^2$ and a point $(x_0, y_0) \in C$ on this level set,

- if
$$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$$
then we can parametrise C by $y = g(x)$ in a neighbourhood of $(x_0, y_0) \in C$;
- if
$$\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$$
then we can parametrise C by $x = h(y)$ in a neighbourhood of $(x_0, y_0) \in C$;
- if
$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$
then such a parametrization is (in general) **not possible**.