Introduction to Manifolds

Lecture 6

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Recollections

Definition (Diffeomorphism) Let

$$f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$$

with U, V domains in $\mathbb{R}^n$. We say that f is a diffeomorphism if f is bijective, that is there exists $f^{-1} : V \to U$, and if $f \in C^1(U, V)$ and $f^{-1} \in C^1(V, U)$.

Theorem (The Inverse Function Theorem in $\mathbb{R}^n$) Let $\Omega \subseteq \mathbb{R}^n$ be open, let $f \in C^1(\Omega, \mathbb{R}^n)$ and let $x_0 \in \Omega$. **Assume that $Df(x_0)$ is invertible.** Then there exists an open neighbourhood U of x_0 such that $f(U)$ is open and

$$f|_U : U \to f(U)$$

is a diffeomorphism, i.e. it has a differentiable inverse $g\colon f(U) \to U$. Furthermore, for this local inverse g of f, we have

$$Dg(f(x_0)) = \big(Df(x_0)\big)^{-1} .$$

A simple example revisited

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. $df(x_0) \neq 0$ is equivalent to $x_0 \neq 0$.
If $x_0 > 0$, we can choose $U = (0, \infty)$, to get a diffeomorphism

$$\begin{array}{rcccl} f & : & (0,\infty) & \to & (0,\infty) \\ & & x & \mapsto & x^2 \\ & & \sqrt{y} & \leftarrow\!\shortmid & y \end{array}$$

If $x_0 < 0$, we can choose $U = (-\infty, 0)$ to get a diffeomorphism

$$\begin{array}{rcccl} f & : & (-\infty,0) & \to & (0,\infty) \\ & & x & \mapsto & x^2 \\ & & -\sqrt{y} & \leftarrow\!\shortmid & y \end{array}$$

A coordinate transformation, revisited

Let $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2$ be given by $(r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$.

We computed before

$$Df(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \quad \text{so} \quad \det Df(r,\varphi) = r > 0.$$

Hence f is invertible **everywhere locally**, by the Inverse Function Theorem.

f is **not globally invertible**: it is 2π-periodic in the φ variable.

In fact it is easy to check (homework!) that $f\colon U \to V$ is a diffeomorphism between the sets

$$U = \left\{(r,\varphi) \mid \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\} \quad \text{and} \quad V = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}.$$

Its inverse $g = f^{-1}\colon V \to U$ is given by

$$g(x,y) = \left(\sqrt{x^2+y^2}, \arctan\frac{y}{x}\right).$$

Implicit Function Theorem from the Inverse Function Theorem

The setup of the Implicit Function Theorem: we write a point of $\mathbb{R}^{k+m}$ as (x, y) with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^m$. We have a C^1 map $f : \Omega \subset \mathbb{R}^{k+m} \to \mathbb{R}^m$.
We assume the $m \times m$ submatrix $D_y f$ of Df at (x_0, y_0) is **invertible**.
Assume also $f(x_0, y_0) = 0$ for simplicity.

We want to find a function g locally such that $f(x, y) = 0$ iff $y = g(x)$.

In order to apply the Inverse Function Theorem, expand f to a function $F : \Omega \subset \mathbb{R}^{k+m} \to \mathbb{R}^{k+m}$ by

$$F(x, y) = (x, f(x, y)).$$

We compute

$$DF = \begin{pmatrix} I & 0 \\ D_x f & D_y f \end{pmatrix}.$$

The invertibility of $D_y f$ at (x_0, y_0) then clearly means that DF is also invertible at (x_0, y_0).

Implicit Function Theorem from the Inverse Function Theorem

From a C^1 map $f : \Omega \subset \mathbb{R}^{k+m} \to \mathbb{R}^m$ we created a C^1 map

$$F : \Omega \subset \mathbb{R}^{k+m} \to \mathbb{R}^{k+m}$$

with DF invertible at (x_0, y_0).
The Inverse Function Theorem now tells us F has a local differentiable inverse

$$h : (x, y) \mapsto (h_1(x, y), h_2(x, y)).$$

We have

$$(x, y) = (F \circ h)(x, y) = (h_1(x, y), (f \circ h)(x, y))$$

so $h_1(x, y) = x$, and hence

$$h(x, y) = (x, h_2(x, y))$$

with $f(x, h_2(x, y)) = (f \circ h)(x, y) = y$.
In particular, $f(x, h_2(x, 0)) = 0$, and we can take

$$g(x) = h_2(x, 0).$$

This proves the existence of the parametrization g asked for in the I.F.T.

Maps of full rank

Question: For $n \geq m$, given a map

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,$$

what is the maximal value of rank $Df(a)$?

Answer: As $Df(a)$ is an $n \times m$ matrix, its rank is at most m.

Simplest example of a map of maximal rank: the **projection** map

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

given by

$$f(x_1, \ldots, x_n) = (x_{n-m+1}, \ldots, x_n).$$

Indeed, $Df(a)$ consists of a lot of zeros and an $m \times m$ identity matrix.

The **level sets** of this map f are just (affine) linear subspaces $\mathbb{R}^{n-m} \subset \mathbb{R}^n$ given by $x_i = c_i$ for $i \geq n - m + 1$.

Simplifying maps of full rank

From our argument for the Implicit Function theorem, we can deduce the following useful fact about general maps which have maximal rank.

Theorem Let $m \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a C^1 function such that $f(a) = 0$ and rank $Df(a) = m$. Then there is an open neighbourhood $V \subset \mathbb{R}^n$ of a and a diffeomorphism $h : U \subset \mathbb{R}^n \to V$ such that

$$(f \circ h)(x_1, \ldots, x_n) = (x_{n-m+1}, \ldots, x_n)$$

is just projection onto the last m coordinates.

Proof After applying a permutation of coordinates (which is a diffeomorphism of $\mathbb{R}^n$), we can assume that the $m \times m$ matrix formed from the m last columns of $Df(a)$ is invertible. Now the proof we saw just now shows the existence of a local diffeomorphism h such that $(f \circ h)(x, y) = y$, as required.

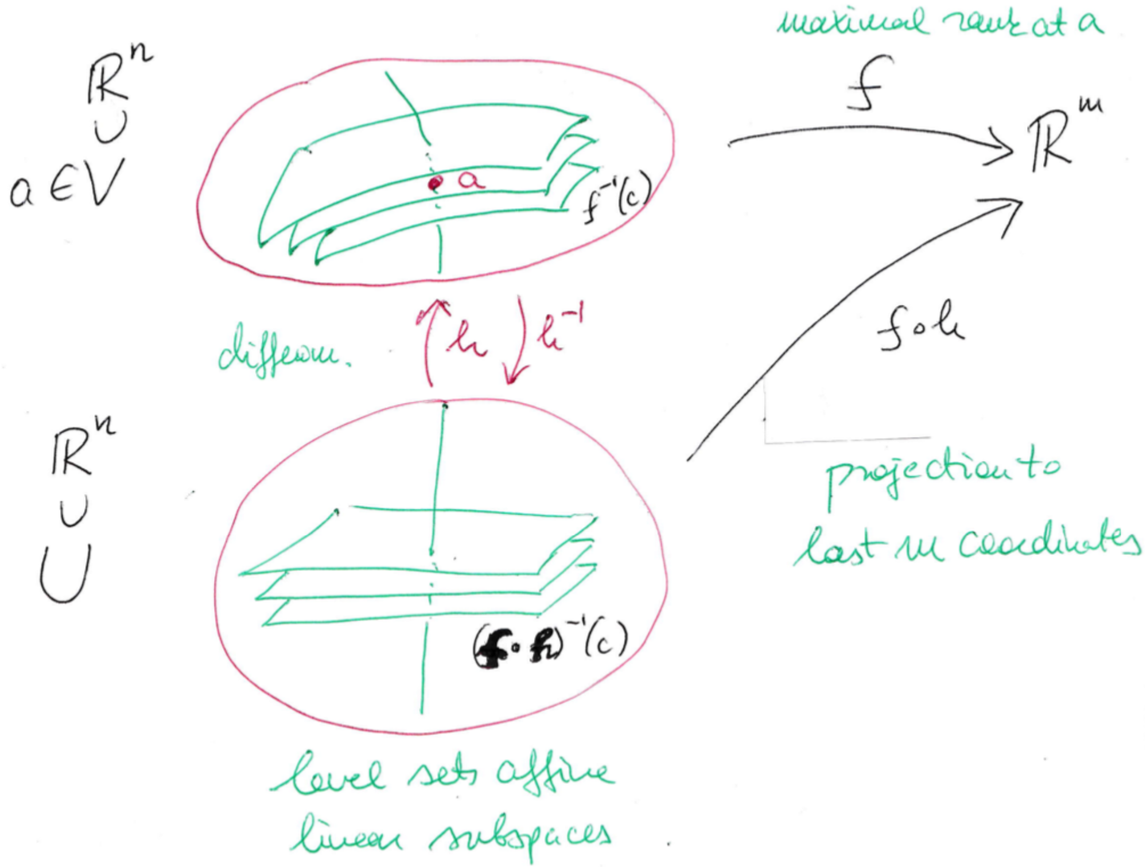
Simplifying maps of full rank

Theorem Let $m \leq n$ and $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ a C^1 function such that $f(a) = 0$ and rank $Df(a) = m$. Then there is an open neighbourhood $V \subset \mathbb{R}^n$ of a and a diffeomorphism $h : U \to V$ such that $f \circ h$ is the projection onto the last m coordinates.

Comments

- Should think of the diffeomorphism appearing in the theorem as a "nonlinear change of coordinates" in the source $\Omega \subset \mathbb{R}^n$.
- What this says is: if $f : \mathbb{R}^n \to \mathbb{R}^m$ and Df has maximal rank at a point of $\mathbb{R}^n$, we can **locally** apply a "change of coordinates" which makes f into the simplest possible rank m map: the projection $\mathbb{R}^{m+k} \to \mathbb{R}^m$.
- In particular, the local structure of level sets of f around points of maximum rank is very simple, up to a "change of coordinates".

Simplifying maps of full rank



Introduction to submanifolds of $\mathbb{R}^n$: hypersurfaces

Let

$$f\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

be a C^1-function, with zero set

$$M = \{x \in \Omega \mid f(x) = 0\} = f^{-1}(0) \subset \mathbb{R}^n.$$

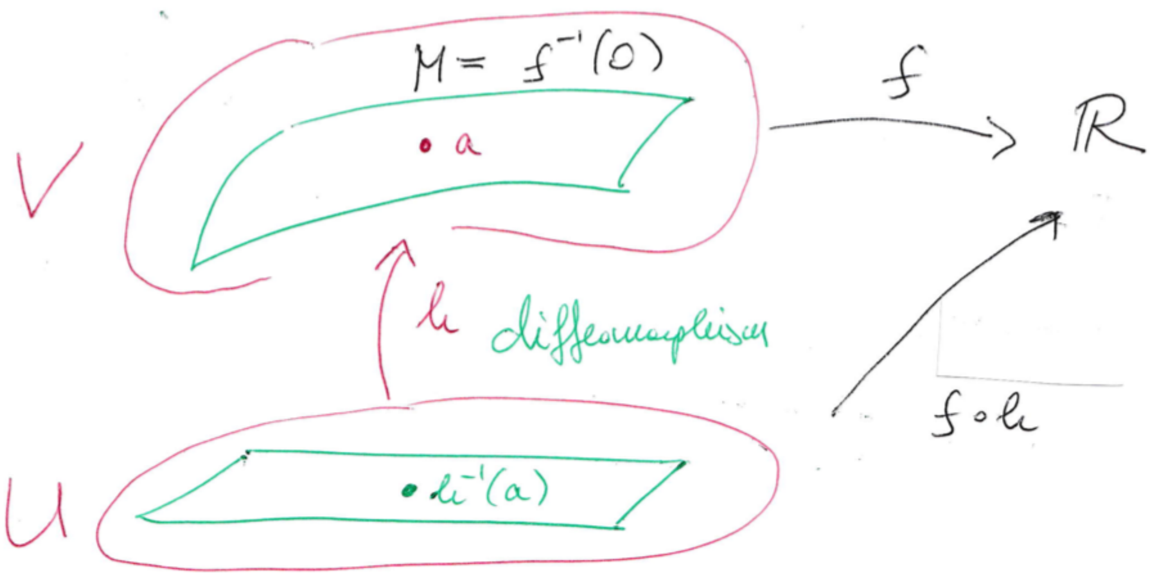
If $Df(a) \neq 0$ for some $a \in M$, then we know from the Implicit Function Theorem that, after a suitable reordering of coordinates, we can represent M in a neighbourhood of a as a graph of a function of $n-1$ variables:

$$x_n = g(x_1, \ldots, x_{n-1}).$$

Equivalently, since f is maximal rank $(= 1)$ at $a \in M$, a neighbourhood of a is diffeomorphic to an open set in $\mathbb{R}^{n-1}$, as, up to a diffeomorphism, f is just the projection to the last coordinate.

Under these conditions, $M \subset \mathbb{R}^n$ is called a **hypersurface** or $n-1$**-dimensional submanifold** of $\mathbb{R}^n$.

Introduction to submanifolds of $\mathbb{R}^n$: hypersurfaces



General submanifolds of $\mathbb{R}^n$

Definition (Submanifolds of $\mathbb{R}^n$) Let $0 < k < n$ be an integer. A set $M \subseteq \mathbb{R}^n$ is called a **k-dimensional submanifold** of $\mathbb{R}^n$, if for every $x_0 \in M$ there exist

- an open neighbourhood Ω of x_0 in $\mathbb{R}^n$ and
- $f \in C^1(\Omega, \mathbb{R}^{n-k})$, such that

$$M \cap \Omega = f^{-1}(0)$$

and

$$\operatorname{rank} Df(x) = n - k \quad \text{for all } x \in \Omega\,.$$

Remark It suffices to require rank $Df(x) = n - k$ for all $x \in M \cap \Omega$, since having maximal rank is an open condition (see the Lecture Notes for details).

A simple example

Let us return to our running example, the unit circle. We have

$$S^1 = \left\{(x_1, x_2) \,|\, x_1^2 + x_2^2 = 1\right\} = f^{-1}(0) \subset \mathbb{R}^2$$

for

$$f(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

Then

$$Df(x_1, x_2) = 2(x_1, x_2)$$

which is nonzero on the domain $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ which contains S^1.

So indeed, S^1 is a one-dimensional submanifold of $\mathbb{R}^2$.

Submanifolds of $\mathbb{R}^n$ are locally graphs of functions

Proposition (Submanifolds can be locally represented as graphs)
For a set $M \subseteq \mathbb{R}^n$, the following properties are equivalent.

(1) M is a k-dimensional submanifold of $\mathbb{R}^n$.

(2) For each $x \in M$ we can, after suitably relabelling the coordinates, write $x = (z_0, y_0)$ with $z_0 \in \mathbb{R}^k$, $y_0 \in \mathbb{R}^{n-k}$ and find an open neighbourhood U of z_0 in $\mathbb{R}^k$, an open neighbourhood V of y_0 in $\mathbb{R}^{n-k}$, and a map $g \in C^1(U, V)$ with $g(z_0) = y_0$, such that

$$M \cap (U \times V) = \{(z, g(z)) \mid z \in U\}.$$

Proof of (1) $\Rightarrow$ (2) After possibly relabelling the coordinates we can write x as $x = (z_0, y_0)$ such that $D_y f(x)$ is invertible. Then property (2) follows from the Implicit Function Theorem.

Submanifolds of $\mathbb{R}^n$ are locally graphs

Proposition (Submanifolds can be locally represented as graphs) For a set $M \subseteq \mathbb{R}^n$, the following properties are equivalent.

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(2) For each $x \in M$ we can, after suitably relabelling the coordinates, write $x = (z_0, y_0)$ with $z_0 \in \mathbb{R}^k$, $y_0 \in \mathbb{R}^{n-k}$ and find an open neighbourhood U of z_0 in $\mathbb{R}^k$, an open neighbourhood V of y_0 in $\mathbb{R}^{n-k}$, and a map $g \in C^1(U, V)$ with $g(z_0) = y_0$, such that

$$M \cap (U \times V) = \{(z, g(z)) \mid z \in U\}.$$

Proof of (2) $\Rightarrow$ (1) Assume that (2) is satisfied. Define $\Omega = U \times V$ and $f \in C^1(\Omega, \mathbb{R}^{n-k})$ via

$$f(z, y) = y - g(z).$$

Then $M \cap \Omega = f^{-1}(0)$ and $Df(z, y) = (-Dg(z), \mathrm{Id}_{n-k})$.

It follows that $\operatorname{rank} Df(z, y) = n - k$.

Revisiting the simple example

Let us look one final time at the circle

$$S^1 = \left\{(x_1, x_2) \,|\, x_1^2 + x_2^2 = 1\right\} = f^{-1}(0) \subset \mathbb{R}^2$$

for $f(x_1, x_2) = x_1^2 + x_2^2 - 1$.
How to write it as a graph?
If $x = (1, 0) \in S^1$, we have

$$S^1 \cap \big((0,2) \times (-1,1)\big) = \left\{(\sqrt{1 - z^2}, z) \,|\, |z| < 1\right\}.$$

Hence, to get the statement in (2) in the last result, we have to relabel (x_1, x_2) as (x_2, x_1).

Exercise: Draw a picture!