A "nice" class of subsets of $\mathbb{R}^n$.

<u>Recall</u>: i) we have linear subspaces of $\mathbb{R}^n$, given by a bunch of linear equ's

$$S = \ker T \subset \mathbb{R}^n, \text{ with } \mathbb{R}^n \xrightarrow{T} \mathbb{R}^{n-k} \text{ linear, surjective}$$

($\dim S = k$) (by Rank-Nullity) (otherwise replace T by $T': \mathbb{R}^n \to \operatorname{im} T$)

ii) affine linear subspaces

$$S = \underline{b} + \ker T \subset \mathbb{R}^n \quad , \quad T: \mathbb{R}^n \to \mathbb{R}^{n-k} \text{ surjective}$$

fixed vector in $\mathbb{R}^n$

$$= T^{-1}(\underline{v}) \text{ for some } \underline{v} \in \mathbb{R}^{n-k} \text{ fixed}$$

iii.) k-dimensional submanifold of $\mathbb{R}^n$ $M \subset \mathbb{R}^n$, such that $\forall x_0 \in M$, $\exists$ neighbourhood $x_0 \in \Omega \subset \mathbb{R}^n$, and $f \in C^1(\Omega, \mathbb{R}^{n-k})$ such that

- $M \cap \Omega = f^{-1}(0)$, or

$$= f^{-1}(\varepsilon) \quad \text{(shift } f\text{)}$$

and

- $\operatorname{rank} Df(x) = n-k$ for $\forall x \in \Omega$.

<u>Simplest class of examples</u> are hypersurfaces, i.e. $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$; $k = n-1$, $n-k=1$, locally they are given by function $f \in C^k(\Omega, \mathbb{R})$ such that $Df(x) \neq \underline{0}$ $\forall x \in \Omega$.

$$Df(x) = \left(\partial f / \partial x_i (x)\right)$$

Ex: any quadratic form q on $\mathbb{R}^n$ will give a conic $\{q(x) = c$ [redacted]$\} \subset \mathbb{R}^n$.

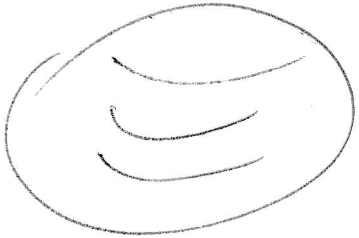
$(c \in \mathbb{R}^n)$

for example: $n = 3$

$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$ gives unit sphere in $\mathbb{R}^3$

$\left(\partial f / \partial x_i \right) = 2(x_1, x_2, x_3)$, nonzero on $\Omega = \mathbb{R}^3 \setminus \{0\}$

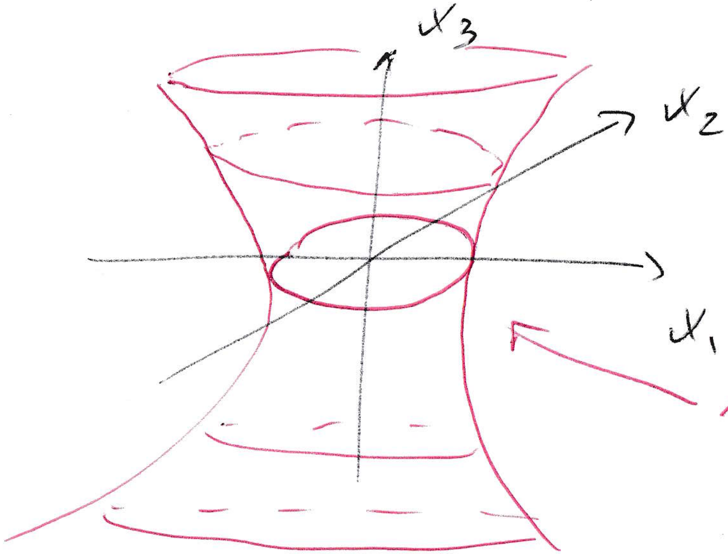
$$\begin{array}{c} \Omega \\ \cup \\ \{f(x) = 0\} \end{array}$$

Get  $S^2 \subset \mathbb{R}^3$, a submanifold.

Q2 : $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 - 1$

$Df(x) = 2(x_1, x_2, -x_3) \neq 0$ on $\mathbb{R}^3 \setminus \{(0,0,0)\} = \Omega$.

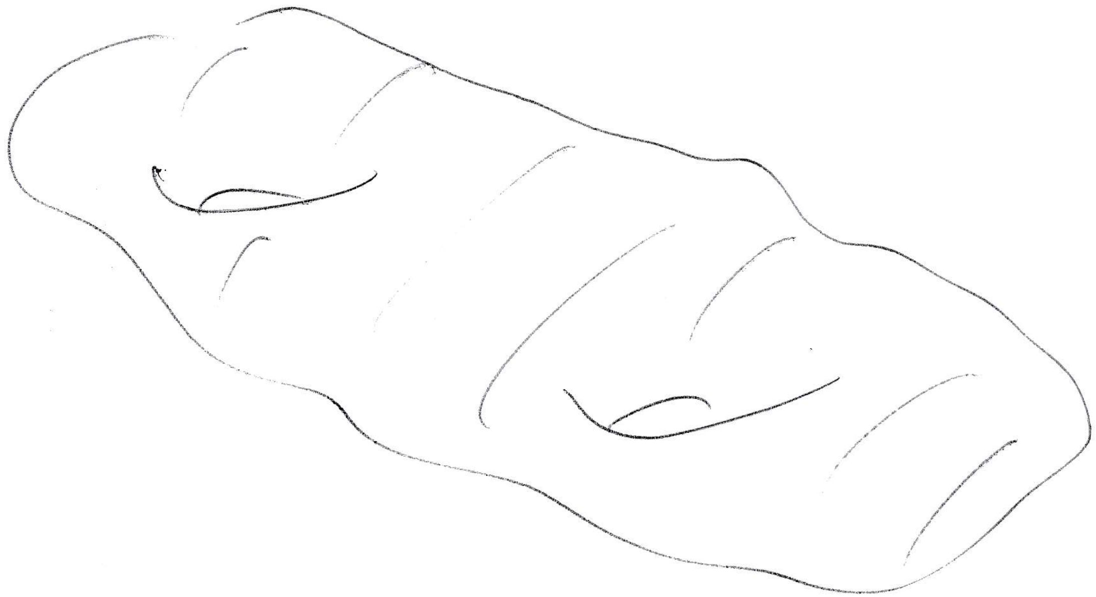
So $f^{-1}(0) \subset \mathbb{R}^3$ is a 2-dimensional submanifold.



hyperboloid

unit circle in (x_1, x_2) plane

The equations do not have to be global!



$\subseteq \mathbb{R}^3$

+ local defining equations

$n=3, k=1$: space curves given by 2 equations (perhaps locally) in $\mathbb{R}^3$
[See an example in Lecture 5, $C = f^{-1}(0,0) \subset \mathbb{R}^3$]

Another set of examples : from matrices (perhaps groups of matrices)

$$O(n) = \{ X \in M_{n\times n}(\mathbb{R}), X^tX = I \} \subset \mathbb{R}^{n^2} = M_{n\times n}(\mathbb{R})$$

Claim : $O(n) \subset \mathbb{R}^{n^2}$ is a submanifold, of dimension $k = n(n-1)/2$

Proof. $O(n) = f^{-1}(0) \subset \mathbb{R}^{n^2}$, where

$$f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$$
$$X \longmapsto X^tX - I$$

– but this won't quite work! Roughly half of these conditions is empty.

Reason: $(X^tX - I)^t = X^tX - I$, so $\boxed{f: \mathbb{R}^{n^2} \longrightarrow \mathrm{Sym}_n(\mathbb{R})}$.

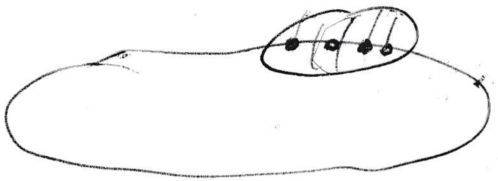
$O(n) = \{f^{-1}(0)\} \subseteq \mathbb{R}^{n^2}$, and we need to check the condition of df having maximal rank.

For $H \subseteq \mathbb{R}^{n^2}$, we have

$$df(X)(H) = H^tX + X^tH \sim (X+H)^t(X+H) - X^tX$$

$df(X): \mathbb{R}^{n^2} \longrightarrow \mathrm{Sym}_n(\mathbb{R})$ - we need this to be surjective for any $X \in O(n)$.

Comment: (rank $Df(x)$ maximal near $x \in M$) is equivalent to (rank $Df(x)$ maximal at all $x \in M$)



$M \subset \mathbb{R}^n$

Last step: $df(X)$ surjective for $X \in O(n)$. Take $Z \in Sym_n(\mathbb{R})$,

let $H = \frac{1}{2}(X \cdot Z)$. Then

$$df(X)(H) = \frac{1}{2} Z^t X^t X + \frac{1}{2} X^t X Z$$

$$= \frac{1}{2} Z^t + \frac{1}{2} Z$$

$$= Z.$$

This proves surjectivity of $df(X)$ and hence the Claim.

$\checkmark$

$\dim Sym_n(\mathbb{R}) = \frac{n(n+1)}{2} = n - k$, giving us the value of k.

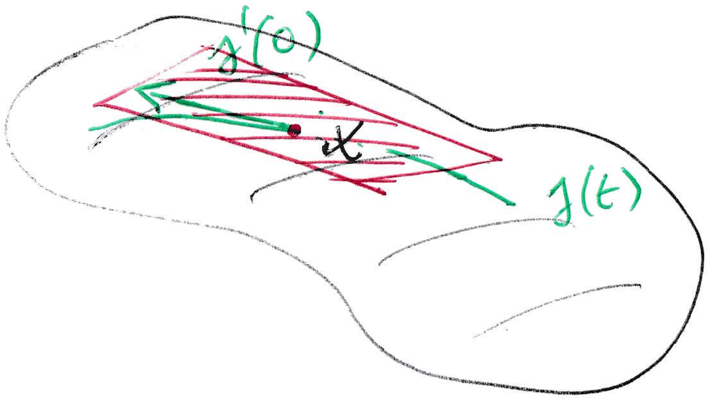
<u>Note</u> i.) $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is an open subset $\det(X) \neq 0$, so a submanifold with $k = n^2$, $n^2 - k = 0$, $f = 0$.

ii.) $SL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is also a submanifold (Problem Sheet)

$$\begin{array}{c} SL_n(\mathbb{R}) \\ \| \\ \{\det X = 1\} \end{array}$$

List of groups that are also submanifolds of $\mathbb{R}^{n^2}$: $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$, ... Lie groups

Linear approximation to a submanifold in $\mathbb{R}^n$ – tangent space



<u>Definition</u> Let $M \subseteq \mathbb{R}^n$ be a k-dimensional submanifold, $x \in M$. We call a vector $v \in \mathbb{R}^n$ a <u>tangent vector to M at $x \in M$</u> if there exists a C^1-function $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ such that

a.) $\gamma(t) \in M$ for all $t \in (-\varepsilon, \varepsilon)$

b.) $\gamma(0) = x$ and $\gamma'(0) = d\gamma(0) = v$; here $d\gamma(0) : \mathbb{R} \longrightarrow \mathbb{R}^n$

Definition The tangent space $T_x M \subset \mathbb{R}^n$ is the set of tangent vectors at x.

(Not clear: this is a linear subspace.)

The normal space $N_x M = (T_x M)^\perp$, the orthogonal to all the tangent vectors in $\mathbb{R}^n$, a linear subspace of $\mathbb{R}^n$ by definition.

Proposition Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold. Then at $x \in M$ there exists a neighbourhood $x \in \Omega \subset \mathbb{R}^n$ and [redacted] $f \in C^1(\Omega, \mathbb{R}^{n-k})$, with $M \cap \Omega = f^{-1}(0)$, Df maximal rank near x. Then

$$T_x M = \ker Df(x)$$

Note This implies in particular that $T_x M$ is a linear subspace of $\mathbb{R}^n$ of dimension k.

Proof Step 1 $T_x M \subseteq \ker Df(x)$

By definition, ~~[illegible]~~ $v \in T_x M$ if $\exists\ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with properties a., - b., above.

a.) $\Rightarrow$ $f(\gamma(t)) = 0$, hence

$$0 = \frac{d}{dt} f(\gamma(t)) = Df(\gamma(t))\,\gamma'(t)$$, so at $t = 0$ get

$$0 = Df(x)(\gamma'(0))$$

i.e. $Df(x)(v) = 0$, $v \in \ker Df(x)$

Step 2 "Generate" lots of tangent vectors - harder, since M is given locally in implicit form $M \cap \Omega = f^{-1}(0)$.

This is where the Implicit Function Theorem comes in handy - it gives M in explicit form.

[Proof to be finished next time...]

$\underline{\text{Ex}}$ of tangent space: $q(\underline{v}) = \underline{v}^t A \underline{v}$, $A \in \mathrm{Sym}_n(\mathbb{R})$, $f(\underline{v}) = q(\underline{v}) - 1$, $M = f^{-1}(0)$

We computed before: $Df(x) = 2(A\underline{x})^t$

If $\det A \neq 0$, then for $\underline{x} \neq 0$, we get $Df(x) \neq 0$.

Then

$$T_x M = \{ v \in \mathbb{R}^n : 2\langle \underline{v}, A\underline{x} \rangle = 0 \} = (Ax)^\perp$$

$$N_x M = \langle A\underline{x} \rangle$$

$\underline{\text{Eg}}$ $A = I$, $\{ f(\underline{v}) = 0 \} \subset \mathbb{R}^n$ is the unit sphere S^{n-1}

$$T_{\underline{x}} S^{n-1} = \langle \underline{x} \rangle^\perp$$

$$N_{\underline{x}} S^{n-1} = \langle \underline{x} \rangle$$