<u>Theorem</u> $M \subset \mathbb{R}^n$ a k-dimensional submanifold, locally given by $f \in C^1(\Omega, \mathbb{R}^{n-k})$ as $f^{-1}(0) = \Omega \cap M$, with rank $Df(x) = n-k$ for $x \in M \cap \Omega$. Then $\forall x \in M \cap \Omega$,

$$T_x M = \ker Df(x)$$

<u>Proof</u> $\subseteq$ ✓

$\supseteq$ We are going to use the Implicit Function Theorem.

After permutation of coordinates, we can assume $x = (z_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and there exist open subsets $z_0 \in U \subset \mathbb{R}^k$, $y_0 \in V \subset \mathbb{R}^{m-k}$ and

$g \in C^1(U, V)$ such that ...

$$(*) \qquad M \cap \underset{\mathbb{R}^n}{\underset{\cap}{(U \times V)}} = f^{-1}(0) = \{(z, g(z)) : z \in U\}$$

implicitly $M = f^{-1}(0)$ | explicitly $M = \{(z, g(z))\}$

This will allow us to write down some explicit tangent vectors.

Let

$$\begin{aligned} G : U &\longrightarrow \mathbb{R}^n \\ z &\longmapsto (z, g(z)) \end{aligned}$$

For $\xi \in \mathbb{R}^{\ell}$, and small ε, set

$$\gamma(t) = G(z_0 + t\xi)$$

Then

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$$

and by $(*)$, $\operatorname{im}(\gamma) \subseteq M \cap (U \times V) \subseteq \mathbb{R}^n$, so we have a piece of a curve (C^1 image of open interval $(-\varepsilon, \varepsilon)$ inside M)

Then

$$\gamma'(t) = DG(z_0 + t\xi)\,\xi \qquad \text{(Chain Rule)}$$

so

$$\boxed{\gamma'(0) = DG(z_0)\,\xi}$$

Here $DG(z_0) : \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^n$ is a linear map.

So we get

$$(**) \quad \boxed{\operatorname{im} DG(z_0) \subseteq T_x M \subseteq \ker Df(x)}$$

However, $DG(z_0) = \begin{pmatrix} I_k \\ Dg(z_0) \end{pmatrix}$ is <u>injective</u>.

$$\dim \operatorname{im} DG(z_0) = k = n - \underbrace{\operatorname{rank} Df(x)}_{n-k} = \dim Df(x)$$

So $(**)$ must be $=$ everywhere.

$\square$

Example $\quad O(n) = \{X^t X = I\} \subseteq \mathbb{R}^{n^2}$ is a submanifold of dimension $\frac{n(n-1)}{2}$

$I \in$

$$f : \mathbb{R}^{n^2} \longrightarrow \mathrm{Sym}_n(\mathbb{R})$$
$$X \longmapsto X^t X - I$$

$$T_I\, O(n) \subseteq \mathbb{R}^{n^2} = \mathrm{Mat}_n(\mathbb{R})$$

$\parallel$

$\ker Df(I) \qquad\qquad Df(X)(H) = X^t H + H^t X$

$\parallel \qquad\qquad Df(I)(H) = H + H^t$

$\mathrm{Skew}_n(\mathbb{R})$

Note $\quad \dim \mathrm{Skew}_n(\mathbb{R}) = \dfrac{(n-1)n}{2} = \dim O(n)$

Question $O(n)$ is - a submanifold of $Mat_n(\mathbb{R})$
- a group ("Lie group")

$$O(n) \times O(n) \to O(n)$$
$$(A, B) \mapsto A \cdot B$$

$Skew_n(\mathbb{R})$ is - tangent space at I to $O(n)$
- "Lie algebra" $[\,,\,]$ Lie bracket

$$[\,,\,] : Skew_n(\mathbb{R}) \times Skew_n(\mathbb{R}) \to Skew_n(\mathbb{R})$$
$$(A, B) \longmapsto [A, B] = AB - BA$$

(See later in Part B-C courses...)

Think about $SL(n, \mathbb{R})$ and its tangent space at I!

Extremization with constraints: the simplest case

Suppose $f, g : C^1(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}^2$

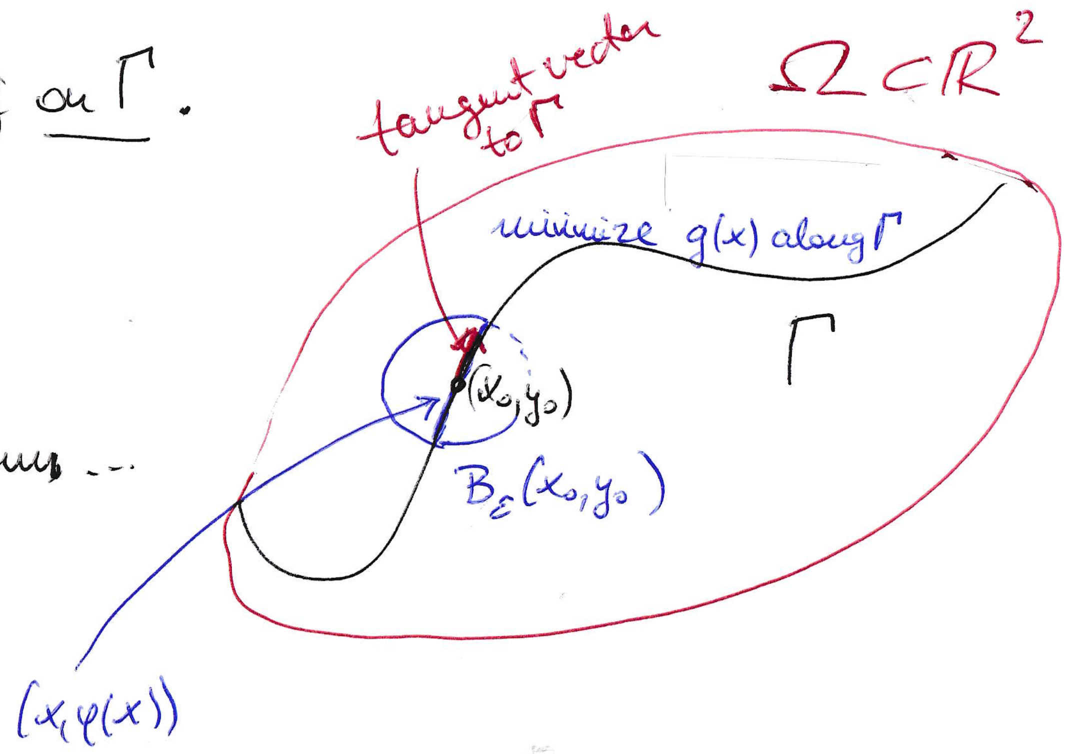
Consider $\Gamma = \{(x,y) \in \Omega : f(x,y) = 0\} = f^{-1}(0) \subset \Omega \subset \mathbb{R}^2$

Pick $(x_0, y_0) \in \Gamma$ such that $\nabla f(x_0, y_0) \neq 0$, then $\Gamma \subset \Omega$ is a one-dimensional submanifold of $\mathbb{R}^2$ near (x_0, y_0).

We want to minimize g on Γ.

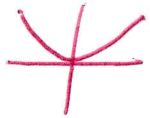
Assume $\partial_y f(x_0, y_0) \neq 0$.

By Implicit Function Thm ...



$$\Gamma \cap B_\varepsilon(x_0,y_0) = \{(x,\varphi(x))\} \quad \text{where } \varphi : I \longrightarrow \mathbb{R}, \quad x_0 \longmapsto y_0$$

<u>Assume</u> that for $(x,y) \in \Gamma \cap B_\varepsilon(x_0,y_0)$, we have $g(x,y) \geq g(x_0,y_0)$ $(*)$

(graph of g looks like  along the curve)

Tangent vector to Γ at x is given by $(1,\varphi'(x))^t$ and we know

$$(1,\varphi'(x))^t \perp \nabla f(x,\varphi(x))$$

$G(x) = (x, g(x))$, then $(*)$ tells us

$$0 \underset{\text{by }(*)}{=} G'(x) \underset{\text{chain rule}}{=} \partial_x g(x_0,y_0) + \partial_y g(x_0,y_0)\cdot \varphi'(x_0)$$

$$= \left\langle \nabla g(x_0, y_0), \begin{pmatrix} 1 \\ \varphi'(x_0) \end{pmatrix} \right\rangle$$

$$\Longrightarrow \quad \nabla g(x_0, y_0) \perp \begin{pmatrix} 1 \\ \varphi'(x_0) \end{pmatrix}$$

$$\boxed{\text{\underline{Conclusion}}\quad \nabla f(x_0,y_0) \text{ and } \nabla g(x_0,y_0) \text{ are proportional at all stationary points, in particular minima/maxima, of } g \text{ along } \Gamma.}$$

Constant of proportionality: "Lagrange multiplier"

__Theorem__ (Lagrange multipliers)

Let $\Omega \subset \mathbb{R}^n$ domain, $f : \Omega \to \mathbb{R}^{n-k}$, $g \in C^1(\Omega, \mathbb{R})$

Assume
- $x_0 \in f^{-1}(0)$ is a local extremum of g on $f^{-1}(0)$ (ie. $\exists$ open ball around x_0 such that if $f(x) = 0$ in the ball, then $g(x) \gtrless g(x_0)$ for a fixed choice of inequality)
- $\nabla f(x_0)$ is of maximal rank $(n-k)$.

__Then__ $\nabla g(x_0) \subseteq \mathrm{Span}\,(\nabla f_i(x_0)) = \mathrm{Im}\,(Df(x_0))$

__ie__ : $\exists\, \lambda_1, \dots, \lambda_{n-k} \in \mathbb{R}$ "the Lagrange multipliers"

such that
$$\boxed{\nabla g(x_0) = \sum_{i=1}^{n-k} \lambda_i \nabla f_i(x_0)}$$

Summary

Analysis

- $f: \mathbb{R}^n \to \mathbb{R}^m$
- differentiability: good linear approximation
  $df(x): \mathbb{R}^n \to \mathbb{R}^m$ linear map
- $df(x) = Df(x)$ Jacobian matrix
  given by partial derivatives of components
- $f \in C^1(\Omega, \mathbb{R}^m)$: all first partials exist & are continuous
  $\Rightarrow$ differentiability of f

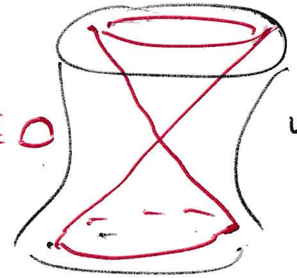
Geometry

- $\{f=0\} \subset \mathbb{R}^n$, $f \in C^1(\Omega, \mathbb{R}^{n-k})$
  maximal rank assumption
  [redacted] Implicit Function Thm
  $\{f=0\} =$
  $=$ graph(g)
  $\updownarrow$ implicit and explicit descriptions
  Inverse Function Thm
  diffeomorphism
- Submanifolds of $\mathbb{R}^n$
  conics / quadrics
  matrix groups

- Chain Rule
- Versions of Intermediate Value Theorem

- Tangent spaces: linear approximation at each point

$x^2+y^2-z^2=0$



$x^2+y^2-z^2=-1$

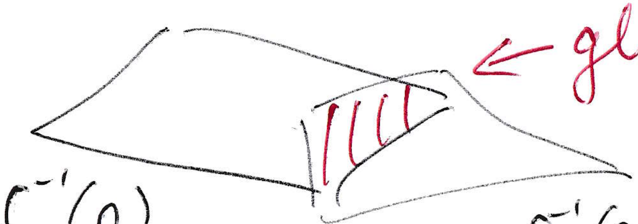
locality

General manifolds

"glued together" from pieces of Euclidean space

Abandon maximum rank assumption

Singularity theory



glue by diffeomorphism

$f_1^{-1}(0) \subset \mathbb{R}^{n_1}$

$f_2^{-1}(0) \subset \mathbb{R}^{n_2}$