Introduction to Manifolds

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szendroi@maths.ox.ac.uk

Example sheet 1

1. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{|xy|^\alpha}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0), \\ \\ 0 \text{ for } (x, y) = (0, 0); \end{cases}$$

where $\alpha > 0$. Find the values of α for which f is

(a) continuous at $(0, 0)$;

(b) differentiable at $(0, 0)$.

2. A function is called *homogeneous of degree* k if $f(\lambda x) = \lambda^k f(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^n$.

(a) Show that if f is homogeneous of degree k, then

$$\langle \nabla f(x), x \rangle = k f(x).$$

(b) Show conversely that if f satisfies this equation, then f is homogeneous of degree k.

3. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} \text{ for } (x, y) \neq (0, 0), \\ \\ 0 \text{ for } (x, y) = (0, 0). \end{cases}$$

Show that all the directional derivatives of f exist at the origin, but f is not differentiable at the origin.

4. In this question we use the Hilbert-Schmidt matrix norm

$$\| A \| = \left(\sum_{i,j} A_{ij}^2 \right)^{\frac{1}{2}}.$$

Show that if H has Hilbert-Schmidt norm less than 1, then $I - H$ is invertible (you may assume that $\| AB \| \leq \| A \| \| B \|$).

5. Let $M_{n\times n}(\mathbb{R})$ denote the vector space of $n \times n$ real matrices. Show that the derivative at the identity of the determinant function

$$\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

is

$$d(\det)_I : h \mapsto \text{trace } h$$

Deduce that the derivative at an arbitrary invertible matrix A is

$$d(\det)_A : h \mapsto \det A.\text{trace } (A^{-1}h).$$

6. (a) Show that the set $GL(n, \mathbb{R})$ of invertible matrices is an open set in $M_{n\times n}(\mathbb{R})$.

   (b) Show that the derivative of the inversion map $\text{Inv } : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is

$$d(\text{Inv})_A : h \mapsto -A^{-1}hA^{-1}$$

   (Hint: look at the case where A is the identity first).