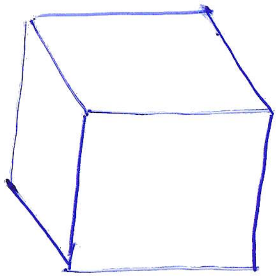
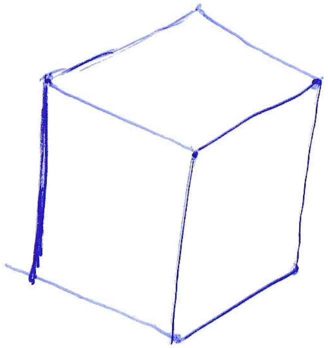
# PROJECTIVE GEOMETRY

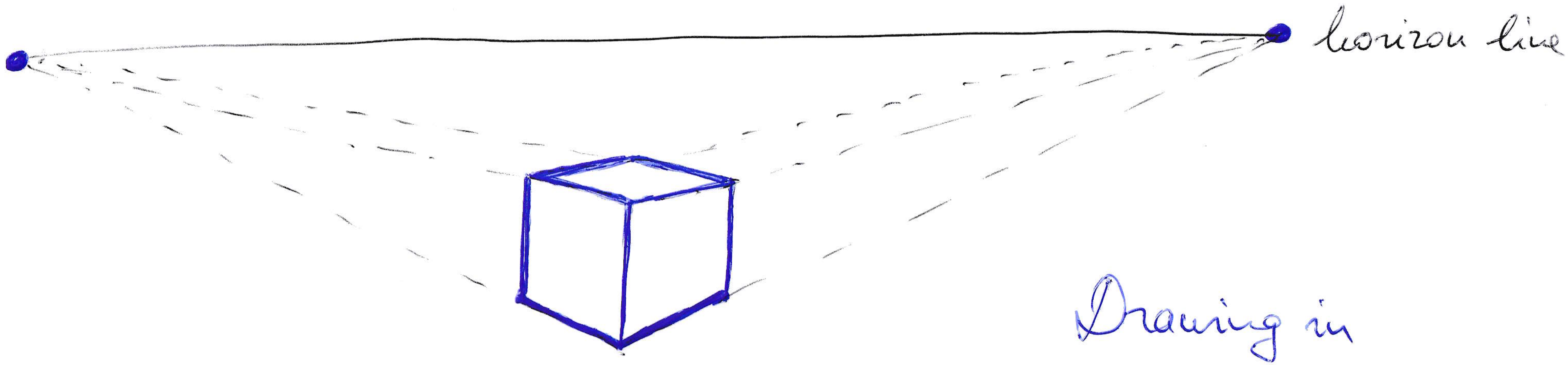
## How to draw a cube?



or



or, perhaps better...



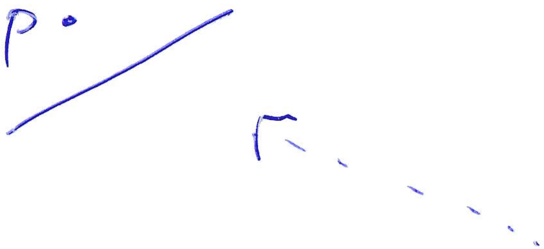

Drawing in perspective

**Key message**: sets of parallel lines get replaced by sets of lines ...

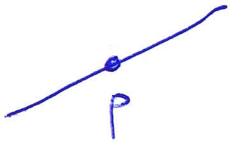
... going through the same point. These points lie on a <u>distinguished line</u>.

# Recall: linear geometry of the (real) plane $\mathbb{R}^2$

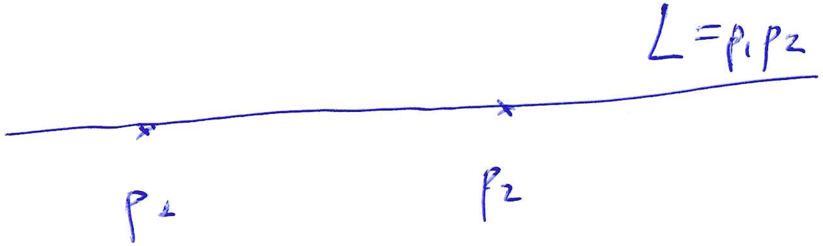
- points $p$ •
- lines $L$



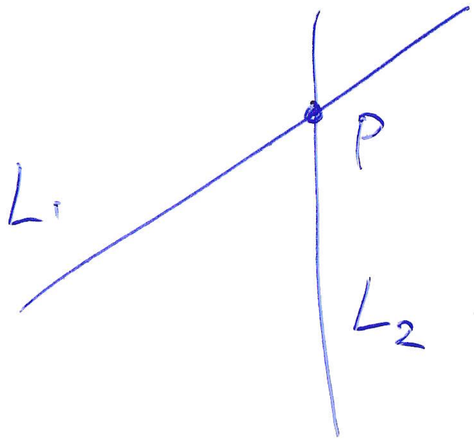
- incidence relation $p \in L$ or $p \notin L$



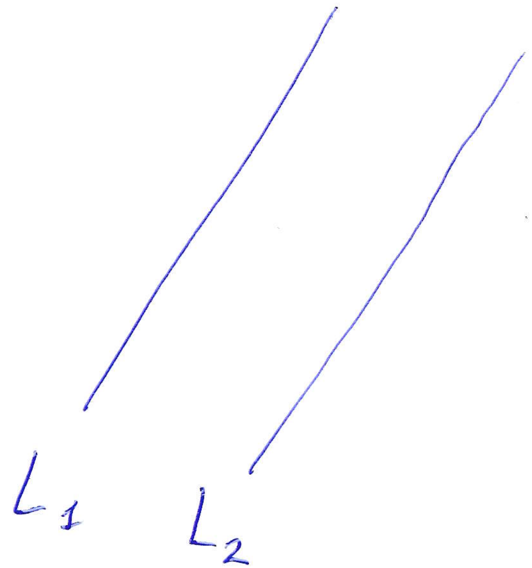
**Property 1** For every two distinct points $p_1 \neq p_2$, there exists a unique line $L$ incident to them / containing them.



__Property 2__ For every two distinct lines $L_1, L_2$, there exists a unique point $p$ incident to them / contained in them, __except__ if $L_1, L_2$ are parallel.




exception:




This is a relation on lines

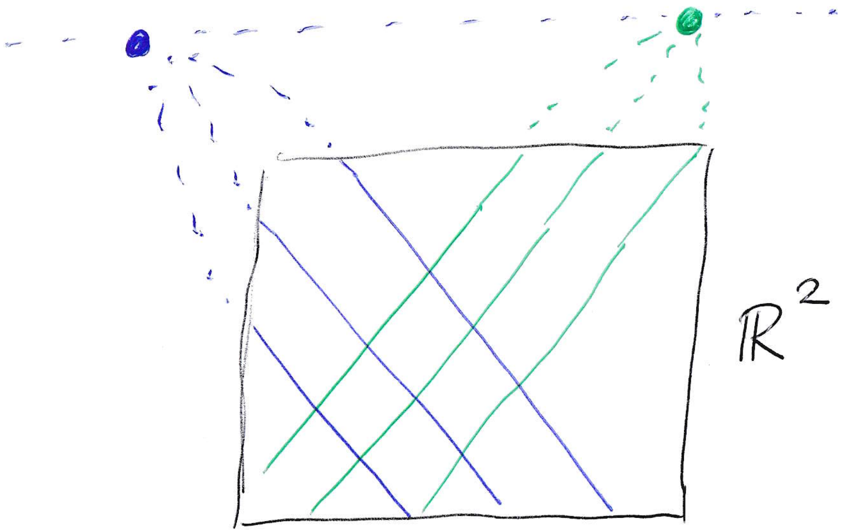
$L_1 \parallel L_2$, $L_1$ parallel to $L_2$, iff $(L_1 \cap L_2 = \emptyset$ or $L_1 = L_2)$

which is an equivalence relation. Equivalence classes are called __directions__.

# What is the projective plane?

Slogan: "parallel lines meet at $\infty$"

$$\mathbb{R}\mathbb{P}^2 = \mathbb{R}^2 \amalg \{\text{one "ideal" point for each direction}\}$$




$\longleftarrow$ ideal points, one for each direction

- the set of ideal points forms the <u>ideal line</u>
- ordinary lines of $\mathbb{R}^2$ are made into the projective <u>line</u> by including their ideal point

Check in this "geometry" Property 1 and Property 2 hold without exceptions!

We are often going to refer to "ordinary" part as affine plane/space

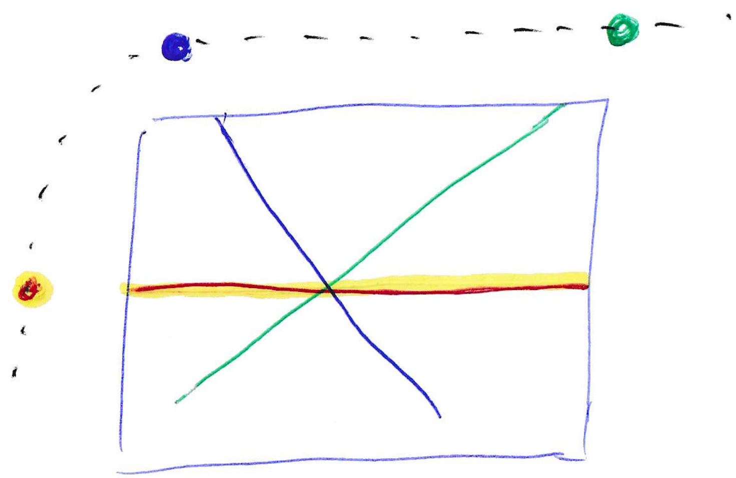
Slogan

$$\text{projective space} = \underset{\text{"finite"}}{\text{affine space}} \amalg \underset{\text{"at } \infty \text{"}}{\text{ideal points}}$$

One final thing

$$\text{ideal line} \longleftrightarrow \text{lines in } \mathbb{R}^2 \text{ through } (0,0)$$

$$\underset{\text{direction}}{\amalg} \text{ ideal point}$$



We work over a general field $\mathbb{F}$.

$$\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p \ldots$$

$V$ finite dimensional vector space over $\mathbb{F}$. Recall: choice of basis sets up an isomorphism $V \cong \mathbb{F}^{\dim(V)}$

<u>Definition</u> Projective space $\mathbb{P}V = \mathbb{P}(V)$ associated to $V$ is the set of one-dimensional linear subspaces (lines) of $V$.

A one-dimensional subspace of $V$ is $\langle \underline{v} \rangle$ for $\underline{v} \in V \setminus \{0\}$.

Two vectors span the same one-dim subspace if and only if they are nonzero constant multiples of each other.

$$\mathbb{P}(V) = V\setminus\{0\} \big/ \underline{v} \sim \lambda \underline{v} \text{ for } \lambda \in \mathbb{F}^* = \mathbb{F}\setminus\{0\}$$

So $\mathbb{P}(V)$ is the set of equivalence classes $[\underline{v}]$ for $\underline{v} \neq 0$.

$$[\underline{v}] = [\underline{w}] \iff \underline{w} = \lambda \underline{v},\ \lambda \in \mathbb{F}^*.$$

$$\dim \mathbb{P}V = \dim V - 1$$

<u>Note</u> $\dim V = 1$ then indeed $\mathbb{P}V = \{pt\}$ one point, $\dim \mathbb{P}V = 0$.

By convention, $\dim \mathbb{P}(V) = -1$ for $V = \{0\}$ $\mathbb{P}V = \emptyset$.

<u>Def</u> A <u>projective linear subspace</u> of $\mathbb{P}(V)$ is $\mathbb{P}(U)$ for a linear subspace $U \subset V$.

$$\left.\begin{array}{llll}
\dim \mathbb{P}(U) = 1 & \iff \dim U = 2 & \iff & \mathbb{P}(U) \text{ projective line} \\
\dim \mathbb{P}(U) = 0 & \iff \dim U = 1 & \iff & \mathbb{P}(U) \text{ projective point} \\
\dim \mathbb{P}(U) = 2 & \iff \dim U = 3 & \iff & \mathbb{P}(U) \text{ projective plane} \\
\dim \mathbb{P}(U) = \dim \mathbb{P}(V) - 1 & \iff \dim U = \dim V - 1 & \iff & \mathbb{P}(U) \text{ a } \underline{\text{hyperplane}}
\end{array}\right\} \text{inside } \mathbb{P}(V)$$

**Lemma 1** Through every two distinct points P and Q of $\mathbb{P}(V)$, there exists a unique (projective) line.

**Proof** $P = [p]$, $p \in V \setminus \{0\}$
$Q = [q]$, $q \in V \setminus \{0\}$
with $p, q$ linearly independent since $P \neq Q$.

Let $U = \langle p, q \rangle \leq V$, a 2-dimensional subspace; clearly $\mathbb{P}(U)$ is the unique projective line containing $P, Q$.

$\square$

**Lemma 2** In a projective plane $\mathbb{P}(V)$, $\dim V = 3$, every two distinct lines $L_1, L_2$ meet in a unique point.

**Proof** $L_1 = \mathbb{P}(U_1)$ $U_1 \leq V$ 2-dimensional
$L_2 = \mathbb{P}(U_2)$ $U_2 \leq V$ - - - -
with $U_1 \neq U_2$

Now by the Dimension of Intersection Formula,

$$\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim \langle U_1, U_2 \rangle \quad \leftarrow \text{linear span}$$

$$= \dim U_1 + \dim U_2 - \dim(U_1 + U_2)$$

$$\qquad \overset{\shortparallel}{2} \qquad\quad \overset{\shortparallel}{2} \qquad\quad \overset{\shortparallel}{-3}$$

$U_+$ not equal, since
$U_1$ $\qquad$ $U_2 \neq U_1$

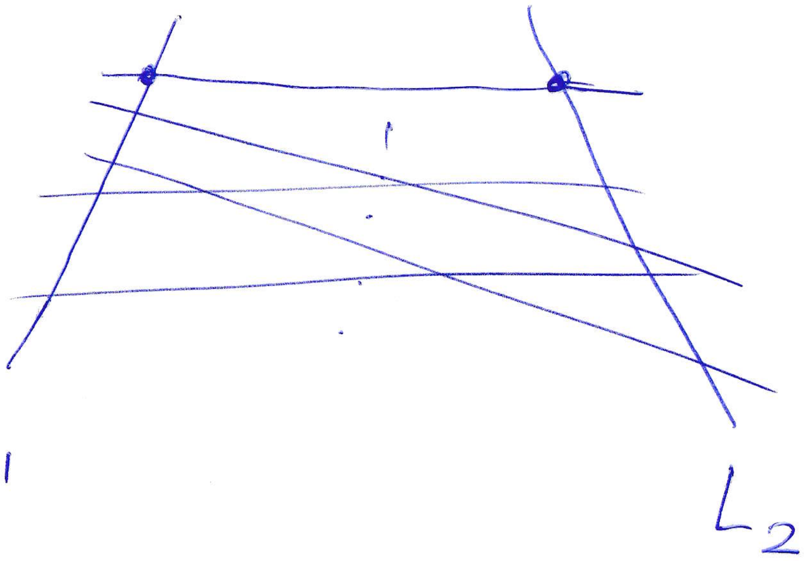
$$= 1$$

and $L_1 \cap L_2 = \mathbb{P}(U_1 \cap U_2) =$ a unique point.

$\square$

## In general

**Definition** Given projective linear subspaces $L_1, L_2$ of $\mathbb{P}(V)$, with $L_i = \mathbb{P}(U_i)$, the projective span $\langle L_1, L_2 \rangle = \mathbb{P}(U_1 + U_2)$ another projective linear subspace of $\mathbb{P}(V)$.




Geometric meaning:
see Problem Sheet 1

**Proposition** (Projective Dimension of Intersection Formula)

Let $L_1, L_2$ be two projective linear subspaces of a projective space $\mathbb{P}V$. Then

$$\dim(L_1 \cap L_2) = \dim(L_1) + \dim(L_2) - \dim(\langle L_1, L_2 \rangle)$$

Here the convention $\dim(\emptyset) = -1$ is in force:

$$L_1 \cap L_2 = \emptyset \iff RHS = -1.$$

**Proof** Write $L_i = \mathbb{P}(U_i)$, then $\langle L_1, L_2 \rangle = \mathbb{P}(U_1 + U_2)$; now $L_1 \cap L_2 = \mathbb{P}(U_1 \cap U_2)$

just use

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \geq 0$$

$$-1 \qquad = \qquad -1 \qquad -1 \qquad +1$$

Now subtract one from each!

$\square$