$V/_{\mathbb{F}}$: finite dimensional $\mathbb{F}$ - vector space

$\mathbb{P}V = \{$ one - dimensional vector subspaces of $V \}$

$= V \setminus \{0\} / \underline{v} \sim \lambda \underline{v}, \ \lambda \in \mathbb{F}^\times$

Fix a basis of V $\{\underline{e}_0, \dots, \underline{e}_n\}$, $\underline{v} = \sum_{i=0}^{n} x_i \underline{e}_i$

dim $V = n+1$, dim $\mathbb{P}V = n$

$\underline{v} \sim \lambda \underline{v} \rightsquigarrow (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$

Denote the equivalence class of this vector by $[x_0 : \dots : x_n]$

These are projective coordinates on $\mathbb{P}V = \mathbb{P}(\mathbb{F}^{n+1}) = \mathbb{F}\mathbb{P}^n$

Rules
- $[x_0 : \dots : x_n] = [\lambda x_0 : \dots : \lambda x_n] \quad \lambda \in \mathbb{F}^*$
- Not all x_i are 0.
- $x_i \in \mathbb{F}$

Take $n=1$: $\mathbb{F}\mathbb{P}^1 = \mathbb{P}(\mathbb{F}^2)$.

A point $p \in \mathbb{F}\mathbb{P}^1$ has coordinates $[x_0 : x_1]$

$[1:\alpha] \longleftarrow\!\shortmid \alpha$

<u>Case 1</u> $x_0 \neq 0$. Then $p = [x_0 : x_1] = [1 : x_1/x_0] \longleftrightarrow \mathbb{F}$

$\mathbb{F} \ni x_1/x_0$ determined by p

<u>Case 2</u> $x_0 = 0$, then $x_1 \neq 0$. $p = [0 : x_1] = [0:1]$. $\longleftrightarrow$ one extra point "∞"

<u>Conclusion</u> $\mathbb{F}\mathbb{P}^1 = \mathbb{F} \sqcup \{\infty\}$

decomposition depends on choice of coordinates and choice of x_0

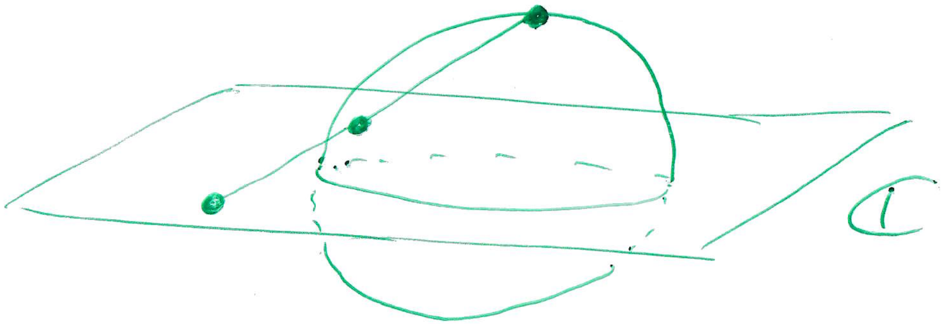
<u>Sub-example</u> Take $F = \mathbb{C}$, then $\boxed{\mathbb{C}P^1 = \mathbb{C} \sqcup \{\infty\}}$

Riemann sphere

<u>Aside</u>:

- $\mathbb{C}P^1$ is a line, so $\boxed{\text{dimension } 1}$, as projective space over $\mathbb{C}$
- $\mathbb{C}P^1$ is a sphere, so dimension 2, in $\mathbb{R}$ coordinates

  ($\operatorname{Re} z$, $\operatorname{Im} z$, and stereographic projection)



Take $n=2$, $\mathbb{F}\mathbb{P}^2 = \mathbb{P}(\mathbb{F}^3)$

$p = [x_0 : x_1 : x_2]$

$$[1 : \alpha : \beta] \longleftarrow\!\!| \ (\alpha, \beta)$$

<u>Case 1</u> $x_0 \neq 0$, $p = [x_0 : x_1 : x_2] = [1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}] \longleftrightarrow \mathbb{F}^2$ $\boxed{\text{finite part}}$

<u>Case 2</u> $x_0 = 0$, $p = [0 : \underbrace{x_1 : x_2}]$ $\longleftrightarrow \mathbb{F}\mathbb{P}^1 = \mathbb{P}(\mathbb{F}^2)$

- one of them nonzero
- up to scale

$\updownarrow$

directions in $\mathbb{F}^2$

$\boxed{\text{ideal points, one for each direction}}$

<u>We get</u>: $\mathbb{F}\mathbb{P}^2 = \mathbb{F}^2 \amalg \mathbb{F}\mathbb{P}^1$

decomposition depends on choices

[redacted]

$= \mathbb{F}$-plane $\amalg$ (ideal points)

Arbitrary n

$\mathbb{F}\mathbb{P}^n = \mathbb{P}(\mathbb{F}^{n+1})$

$x_0 \neq 0 \quad [x_0 : \dots : x_n] = [1 : \underbrace{x_1/x_0 : \dots : x_n/x_0}_{\text{affine coordinates } (\alpha_1, \dots, \alpha_n)}] \longleftrightarrow \mathbb{F}^n$ finite pts

$x_0 = 0 \quad [x_0 : \dots : x_n] = [0 : x_1 \dots : x_n] \longleftrightarrow \mathbb{P}(\mathbb{F}^n)$ ideal pts

$\updownarrow$

directions in $\mathbb{F}^n$

<u>Recall</u> projective linear subspaces in $\mathbb{P}V$ are $\mathbb{P}U$, for $U \leq V$ linear subspace. One way to view these is as 0-locus linear, homogeneous equations

$$U = \left\{ \underline{v} : \sum_{i=0}^{n} \alpha_{ji} x_i = 0 \ , \ j = 1, \dots, m \right\} = \ker(T)$$

with $T : V \to \mathbb{F}^m$

given by (α_{ji})

Can write directly

$$\mathbb{P}U = \left\{ [x_i] : \underbrace{\sum_{i=0}^{n} \alpha_{ji} x_i = 0}_{\text{makes sense up to scale}} \ , \ j = 1, \dots, m \right\} \subseteq \mathbb{P}V$$

__Example__ start with plane $\mathbb{F}^2$

$$\mathbb{FP}^2 = \mathbb{F}^2 \sqcup \mathbb{FP}^1$$

$$\begin{aligned}
[1:x:y] &\leftarrow\!\shortmid (x,y) \\
[0:x_1:x_2] &\longleftarrow\!\shortmid [x_1:x_2] \\
[x_0:x_1:x_2] &\longmapsto [1:\tfrac{x_1}{x_0}:\tfrac{x_2}{x_0}] \qquad x_0 \neq 0 \\
&\longmapsto [x_1:x_2] \qquad x_0 = 0
\end{aligned}$$

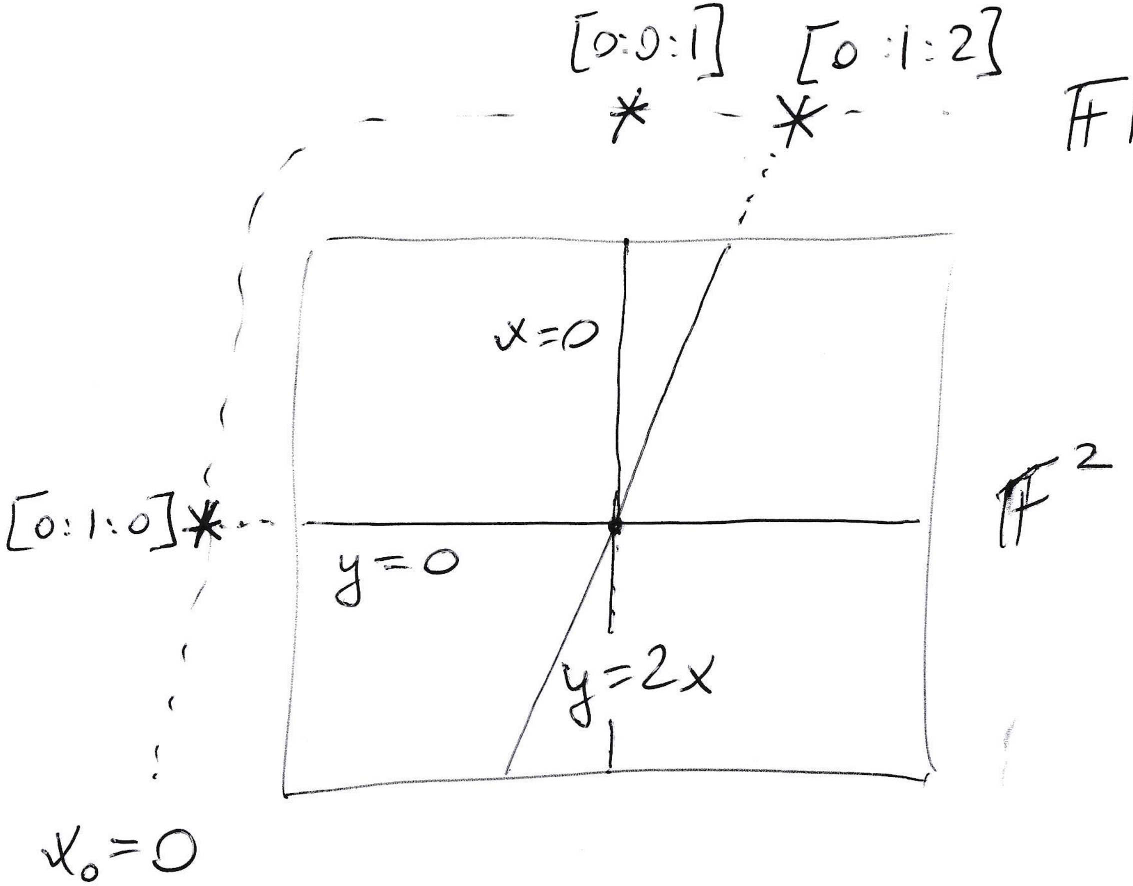
Consider line $\{y=2x\} \subseteq \mathbb{F}^2$

To get the corresponding <u>projective</u> line, use projective coordinates and $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$

Then we get the equation $\{x_2 = 2x_1\} \subseteq \mathbb{FP}^2$.

What solutions are there to this equation? $x_0 \neq 0$: get $\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ on $\{y=2x\}$

$x_0 = 0$: get $[0:1:2] \in \mathbb{FP}^2$



$\mathbb{FP}^1$ "line at ∞" = ideal line

One special line

$L_\infty = \{ x_0 = 0 \} \subseteq \mathbb{FP}^2$

ideal line - not the completion of any line in $\mathbb{F}^2$

$y = mx + b$ or $x = c$

slope $\in \mathbb{F}$ slope ∞

Aside: let $\mathbb{F} = \mathbb{R}$

$$\mathbb{R}\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1}) = \underline{v} \in \mathbb{R}^{n+1} \setminus \{0\} \big/ (\underline{v} \sim \lambda \underline{v},\ \lambda \in \mathbb{R}^*)$$

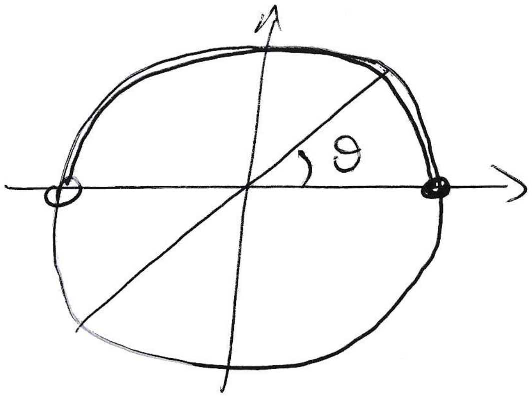
$$= v \in S^n \big/ (\underline{v} \sim \pm \underline{v})$$

where $S^n = \{ v \in \mathbb{R}^{n+1} : |v| = 1 \}$ is the unit n-sphere in $\mathbb{R}^{n+1}$.

Slogan: $\mathbb{R}\mathbb{P}^n = S^n$ with antipodal points identified.

Fact $\mathbb{RP}^n = S^n/(\underline{v} \sim -\underline{v}) = S^n/C_2$ acting by $\underline{v} \mapsto -\underline{v}$

Example 1 $n=1$ $\mathbb{RP}^1 = S^2/(\pm 1)$



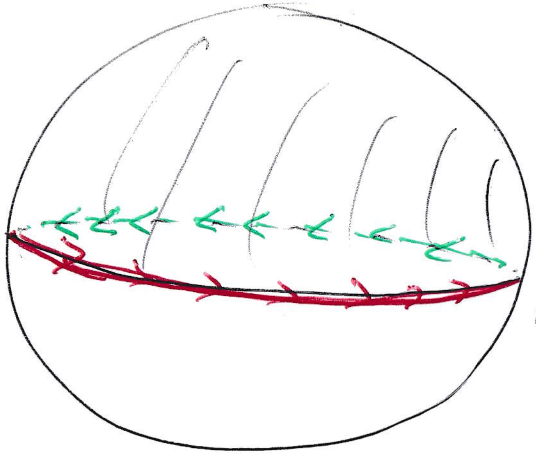
θ and $\theta + \pi$ are different points of S^1 but correspond to the same direction

Actually $\mathbb{RP}^1 \sim S^1$

Proposition S^n is compact (closed and bounded in $\mathbb{R}^{n+1}$), so $\mathbb{RP}^n$ is also compact, as the surjective image of S^n.

Example 2 get $\mathbb{RP}^2 = S^2/(\pm 1) = \cup S^2 \Big/ \left(\begin{array}{l}\text{identification } \underline{v} \mapsto -\underline{v} \\ \text{on the boundary}\end{array}\right)$

"upper" hemisphere $z \geq 0$



identification on boundary as indicated

Note Key point for equations of linear subspaces was that we used linear, homogeneous equations.

We can define interesting subspaces of $\mathbb{F}P^n$ by using homogeneous polynomial equations of (x_i) : total degree of each term is the same.

Ex: $x_1^2 + x_2^2 + 3x_3^2 + x_1 x_2 + x_2 x_3$: homogeneous of degree 2

$x_0^3 + x_1^3 + x_2^3 + x_0 x_1 x_2$: - - - - - 3

Setting these to 0 makes sense in projective space.

$x_0 + x_1^2 + x_2^3$ $\quad\times$

$x_0 x_1 + x_2^3$ $\quad\times$

not homogeneous.