Projective transformations

Recall: for f.d. $\mathbb{F}$-vector space V, $\mathbb{P}V = V \setminus \{0\} / \underline{v} \sim \lambda \underline{v},\ \lambda \in \mathbb{F}^*$

Q: What are reasonable maps/functions $\mathbb{P}V \longrightarrow \mathbb{P}W$?

A: Consider linear maps $T: V \longrightarrow W$

We would like to use T to define

$$\begin{aligned} \tau: \mathbb{P}V &\longrightarrow \mathbb{P}W \\ [\underline{x}] &\longmapsto [T\underline{x}] \end{aligned}$$

If $\underline{v} \sim \lambda \underline{v}$ then $T(\lambda \underline{v}) = \lambda T(\underline{v})$ so τ is well-defined on equivalence classes. But we must also have $T\underline{v} \neq 0$ for $\underline{v} \neq 0$.

Definition To an *injective* linear map $T: V \to W$, we can associate the *projective map* $\tau: \mathbb{P}V \to \mathbb{P}W$

$$[\underline{v}] \longmapsto [T\underline{v}]$$

For $V = W$, an *invertible* linear map $T: V \to V$ gives a *projective transformation* $\tau: \mathbb{P}V \to \mathbb{P}V$

$$[\underline{v}] \longmapsto [T\underline{v}]$$

Note: • a projective transformation is automatically invertible:

τ^{-1} is given by T^{-1}!

• identity projective transformation comes from Id_V

• Composition (of projective transformations) is associative.

<u>Thus</u> to every projective space $\mathbb{P}V$, we have associated its <u>group</u> of projective transformations $PGL(V) = \{ \tau : \mathbb{P}V \longrightarrow \mathbb{P}V \}$

<u>Example</u> $\mathbb{F}P^1 = \mathbb{P}(\mathbb{F}^2)$. T is given by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ invertible.

$$T(\underline{v}) = A\underline{v}$$

$$T(x_0, x_1) = (ax_0 + bx_1,\ cx_0 + dx_1)$$

$$\tau([x_0 : x_1]) = [ax_0 + bx_1 : cx_0 + dx_1]$$

or, assuming $x_1,\ cx_0 + dx_1 \neq 0$, can write this as

$$\tau\left(\left[\frac{x_0}{x_1} : 1\right]\right) = \left[\frac{ax_0 + bx_1}{cx_0 + dx_1} : 1\right] = \left[\frac{a\frac{x_0}{x_1} + b}{c\frac{x_0}{x_1} + d} : 1\right]$$

So if we write $\mathbb{P}(\mathbb{F}^2) = \underset{(x_1 \neq 0)}{\mathbb{F}} \sqcup \underset{[1:0]}{\{\infty\}}$

and use the coordinate $z = x_0/x_1$ on this copy of $\mathbb{F}$, get

$$\boxed{\tau(z) = \frac{az+b}{cz+d}} \quad \text{for } z \in \mathbb{F}$$

Möbius transformations

Upshot: projective transformations acting on $\mathbb{F}P^1$ correspond to Möbius transformations on $\mathbb{F} \cup \{\infty\}$.

(seen before for $\mathbb{F} = \mathbb{C}$)

Next, take $\mathbb{F}P^2 = P(\mathbb{F}^3) = \mathbb{F}^2 \amalg \mathbb{F}P^1$

"finite" "line at ∞"

"ideal line"

$(x_0 \neq 0)$ $(x_0 = 0)$

$T : \mathbb{F}P^2 \longrightarrow \mathbb{F}P^2$ given by 3×3 invertible matrix over $\mathbb{F}$

One can look at the subgroup preserving the line at ∞.

<u>Fact</u> : such projective transformations restrict to <u>affine transformations</u>

$\tau(\underline{z}) = A\underline{z} + \underline{b}$, A : 2×2 matrix over $\mathbb{F}$, invertible

$\underline{b} \in \mathbb{F}^2$ (translation)

for $\underline{z} \in \mathbb{F}^2 \subset \mathbb{F}P^2$.

(See Problem Sheet 1)

Just as $\mathbb{P}V = V \setminus \{0\} / \sim$, we also have

<u>Claim</u> $PGL(V) = GL(V) / T \sim \lambda T, \ \lambda \in \mathbb{F}^*$

$GL(V)$: invertible linear maps $V \to V$

$T \sim \lambda T$: rescaling

<u>Proof</u> Clearly $[T\underline{v}] = [(\lambda T)\underline{v}] = [\lambda T \underline{v}]$ as $\lambda \in \mathbb{F}^*$.

So τ attached to T is the same as τ attached to λT.

Conversely, if T_1, T_2 give the same projective map, then

$$[T_1 \underline{v}] = [T_2 \underline{v}] \text{ for all } \underline{v} \in V \setminus \{0\}$$

Take $\underline{v}, \underline{w}$ linearly independent.

$$T_1 \underline{v} = \lambda T_2 \underline{v}$$

$$T_1 \underline{w} = \mu T_2 \underline{w}$$

$$T_1 (\underline{v} + \underline{w}) = \nu \, T_2 (\underline{v} + \underline{w})$$

Get $\quad 0 = (\lambda - \nu)\, T_2(\underline{v}) + (\mu - \nu)\, T_2(\underline{w})$

$\Longrightarrow \quad \lambda = \mu = \nu$

$\Longrightarrow \quad \boxed{\forall\, \underline{v}, \quad T_1 \underline{v} = \lambda\, T_2 \underline{v} \text{ for a fixed constant } \lambda \in \mathbb{F}^*}$

$\square$

Aside (for group theorists!) $V/\mathbb{F}$

$$GL(V) \text{ group}, \quad Z(GL(V)) = \{\lambda \cdot I : \lambda \in \mathbb{F}^\times\}$$

$$PGL(V) = GL(V)/_{\text{rescaling}} = GL(V)/Z(GL(V))$$

Eg: for $\mathbb{F} = \mathbb{F}_p$ or $\mathbb{F}_q$ $(q = p^n)$ finite, we get very interesting finite groups

$$PGL(n, q) = PGL(\mathbb{F}_q^{\,n})$$

eg. one of them, $PGL(2,7)$ has order 168 and is finite simple.

[Correction: this is not quite right, need to take $PSL(2,7)$, imposing also $\det = 1$]

Recall: the Möbius group acts "triply transitively" on $\mathbb{C} \cup \{\infty\}$

Same proof: ——— " ——— on $\mathbb{FP}^1$ over any field.

Fact Given two sets of three distinct points in $\mathbb{FP}^1$, there is a unique Möbius transformation mapping the first triple to the second one (in fixed order)

Generalization: given {n+2 ... (condition)} in $\mathbb{FP}^n$, there is a unique projective transformation mapping ...

Definition An $(n+2)$-tuple of points $p_0, \dots, p_{n+1} \in \mathbb{FP}^n$ is said to be in general position if either $p_i = [\underline{v}_i]$, each $(n+1)$ subset of $\{\underline{v}_0, \dots, \underline{v}_{n+1}\}$ is linearly independent in $\mathbb{F}^{n+1}$

or $\langle p_{i_0}, \dots, p_{i_n} \rangle = \mathbb{FP}^n$ for each $(n+1)$-subset $\{i_0, \dots, i_n\} \subset \{0, \dots, n+1\}$

(equivalent conditions)

Another alternative formulation: no $(n+1)$-subset of $\{p_0, -, p_{n+1}\}$ lies in a hyperplane of $\mathbb{F}\mathbb{P}^n$!

<u>Theorem</u> Given two sets of $(n+2)$ points $\{p_0, -, p_{n+1}\}$ and $\{q_0, -, q_{n+1}\}$ in $\mathbb{F}\mathbb{P}^n$, both sets in general position, then there exists a unique projective transformation $\tau \in PGL(\mathbb{F}^{n+1})$ such that $\tau(p_i) = q_i$ for all i.

<u>Proof</u> a.) $p_i = [\underline{v}_i]$; for $i = 0, .., n$ get $\underline{v}_0, .., \underline{v}_n$ linearly independent so form a basis (by general position!)

b.) $\underline{v}_{n+1} = \sum_{i=0}^{n} \lambda_i \underline{v}_i$; then each $\lambda_i \neq 0$. Otherwise, this would be a linear dependence relation between $(n+1)$ of the $\underline{v}_i$'s.

c.) Now rescale $\underline{v}_i$ by $\lambda_i \neq 0$, so we can assume $p_i = [\underline{v}_i]$ for $i = 0, .., n$ and $p_{n+1} = [\sum_{i=0}^{n} \underline{v}_i]$.

d.) Similarly, can do the same for the q's: $\exists$ basis $\underline{w}_0, \dots, \underline{w}_n$ such that $q_i = [\underline{w}_i]$, $i = 0, \dots, n$; $q_{n+1} = [\sum_{i=0}^{n} \underline{w}_i]$

e.) Let T be the invertible linear map $\underline{v}_i \longmapsto \underline{w}_i$ for $i = 0, \dots, n$. Then $T(\sum \underline{v}_i) = \sum \underline{w}_i$ so for all $i \in \{0, \dots, n\}$

$$\tau(p_i) = \tau([\underline{v}_i]) = [\underline{w}_i] = q_i$$

and

$$\tau(p_{n+1}) = \tau([\sum \underline{v}_i]) = [\sum \underline{w}_i] = q_{n+1}.$$

Finally for uniqueness (up to scale for T), see notes.

$\square$