Recall that $p_0, p_1, \dots, p_{n+1} \in \mathbb{F}\mathbb{P}^n$ are in general position if

a.) vectors representing any $(n+1)$ - subset are linearly independent in $\mathbb{F}^{n+1}$, equivalently

b.) if each $(n+1)$ of them spans $\mathbb{F}\mathbb{P}^n$.

Theorem The group $PGL(\mathbb{F}^{n+1})$ acts simply transitively on $(n+2)$-tuples of points in general position.

ie: if $\{p_0, \dots, p_{n+1}\}$ respectively $\{q_0, \dots, q_{n+1}\}$ are in general position, then $\exists$ unique $\tau \in PGL(\mathbb{F}^{n+1})$ such that $\tau(p_i) = q_i$.

Remark i) $n=1$: any triple of points (distinct) on $\mathbb{F}P^1$ is projectively equivalent to any other triple

ii) $n=2$: any two proper quadrangles (no three points on a line) are projectively equivalent.

The proof of the general position theorem (given in the last lecture) gives us "good representatives" for set of points in $\mathbb{F}P^n$.

Corollary Given $(n+2)$ points in $\mathbb{F}P^n$ in general position, we can choose a coordinate system so that our points have coordinates

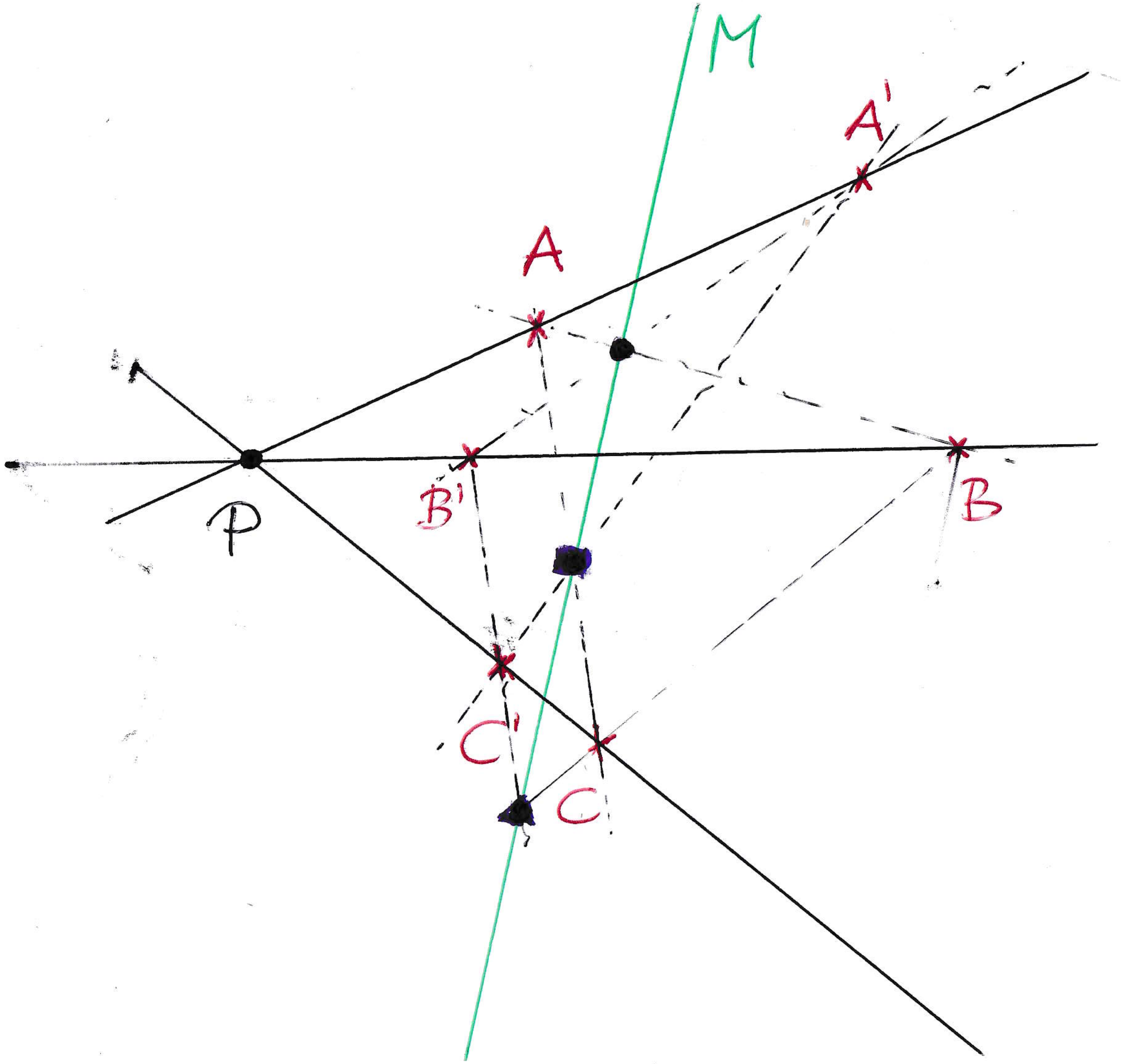
$$p_0 = [1:0:\ldots:0]$$
$$p_1 = [0:1:\ldots:0]$$
$$\vdots$$
$$p_n = [0:\ldots:0:1]$$
$$p_{n+1} = [1:1:\ldots:1]$$

Some classical theorems

Theorem (Desargues) Let P, A, A', B, B', C, C' be points in $\mathbb{F}\mathbb{P}^n$, such that the lines AA', BB', CC' are all distinct, and all meet at the point P.

Then the points of intersection $AB \cap A'B'$, $AC \cap A'C'$, $BC \cap B'C'$ are collinear.

Claim: green line M exists.



One possible proof.

Step 1: assume $n \geq 3$, analyse configurations of points in the picture – they define planes in $\mathbb{FP}^n$ with common points; can deduce they intersect in green line M

Step 2 "lift" configuration from 2d to 3d

Proof Since A, A', P are collinear, the three points have representing vectors $\underline{p}, \underline{a}, \underline{a}'$ such that $\underline{p} = \underline{a} + \underline{a}'$ (by adjusting coefficients).

Similarly, B, B'; C, C' have representing vectors so that, moreover,

$$\underline{p} = \underline{b} + \underline{b}'$$

$$\underline{p} = \underline{c} + \underline{c}'$$

Now $\underline{a} + \underline{a}' = \underline{b} + \underline{b}'$ so $\underline{a} - \underline{b} = \underline{b}' - \underline{a}'$ so $[\underline{a} - \underline{b}] = [\underline{b}' - \underline{a}']$ must lie on the AB line; also on the $A'B'$ line. So we get

$$[\underline{a} - \underline{b}] = [\underline{b}' - \underline{a}'] \in AB \cap A'B' \quad \bullet$$

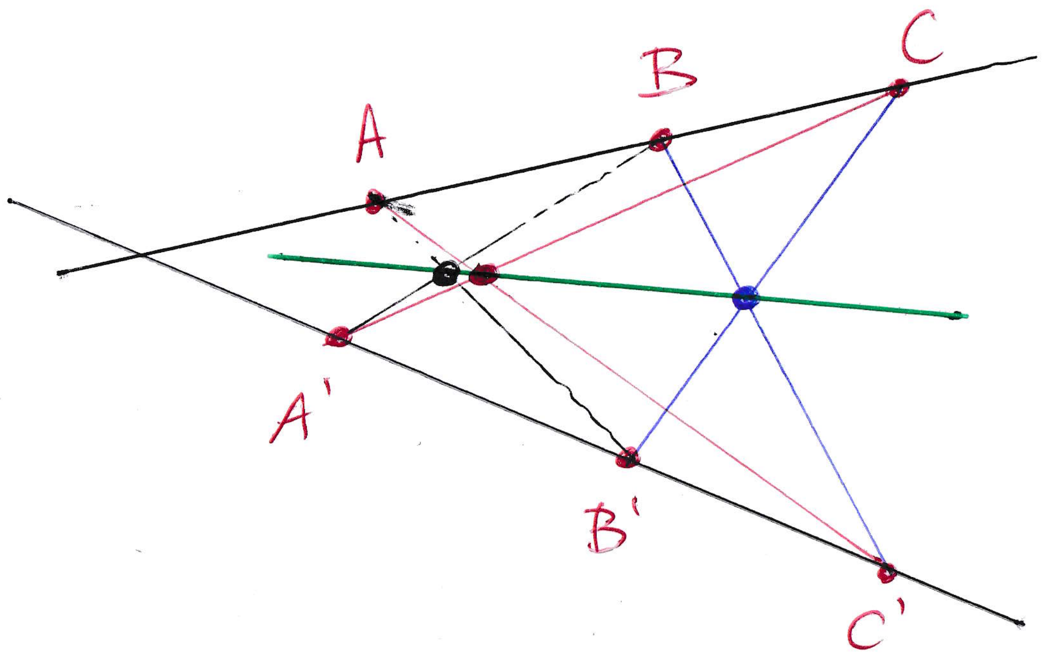
Similarly

$$[\underline{a} - \underline{c}] = [\underline{c}' - \underline{a}'] \in AC \cap A'C' \quad \blacksquare$$

$$[\underline{b} - \underline{c}] = [\underline{c}' - \underline{b}'] \in BC \cap B'C' \quad \blacktriangle$$

Finally $(\underline{a} - \underline{b}) + (\underline{b} - \underline{c}) + (\underline{c} - \underline{a}) = 0$ so these three points all lie on a line. $\square$

The Theorem of Pappus Let A, B, C, respectively A', B', C' be collinear triples of distinct points in $\mathbb{F}P^2$. Then the three points $AB' \wedge A'B$, $AC' \wedge A'C$, $BC' \wedge B'C$ are collinear.



Proof On problem sheet, based on calculations, using initial simplification

Cross-ratio

Let P_0, P_1, P_2, P_3 be points on a projective line $\mathbb{F}\mathbb{P}^1$.

Write $\quad P_i = [\xi_i : \eta_i]$ for $\quad \xi_0, \dots, \eta_3 \in \mathbb{F}$

$$(P_0 P_1 : P_2 P_3) = \frac{\xi_0 \eta_2 - \eta_0 \xi_2}{\xi_0 \eta_3 - \xi_3 \eta_0} \; \frac{\xi_1 \eta_3 - \xi_3 \eta_1}{\xi_1 \eta_2 - \xi_2 \eta_1} \in \mathbb{F}$$

Note this is well-defined as a function of P_0, P_1, P_2, P_3; note $\xi_i \eta_j - \xi_j \eta_i \neq 0$

Also **Lemma** $[P_0 P_1 : P_2 P_3]$ is invariant under $PGL(\mathbb{F}^2)$

Proof $T \in PGL(\mathbb{F}^2)$ comes from $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{F}^2)$

Also $\begin{pmatrix} \xi_i & \xi_j \\ \eta_i & \eta_j \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \xi_i & \xi_j \\ \eta_i & \eta_j \end{pmatrix}$ (both column vectors get multiplied)

So $(\xi_i \eta_j - \xi_j \eta_i) \longmapsto \underbrace{\det T}_{ad-bc \neq 0} (\xi_i \eta_j - \xi_j \eta_i)$

Hence $[P_0 P_1 : P_2 P_3]$ remains unchanged.

$\square$

Theorem Two quadruples of points on $\mathbb{FP}^1$ are projectively equivalent if and only if their cross-ratios are equal.

ie: P_0, P_1, P_2, P_3 distinct; Q_0, Q_1, Q_2, Q_3 distinct points on $\mathbb{FP}^1$

$$\exists\, \tau \in PGL(\mathbb{F}^2) \text{ st } \tau(P_i)=Q_i \iff [P_0P_1 : P_2P_3] = (Q_0Q_1 : Q_2Q_3)$$

Proof $\Rightarrow$ is already done.

$\Leftarrow$ P_0, P_1, P_2, P_3 distinct implies that there exists a coordinate system in which $P_0 = [1:0]$, $P_1 = [0:1]$, $P_2 = [1:1]$ and $P_3 = [\zeta_3 : \eta_3]$

(If we consider the coordinate $\frac{\eta}{\zeta}$ then these are $0, \infty, 1$, and arbitrary $\frac{\eta_3}{\zeta_3}$.)

Now $\xi_3 \neq 0$, $\eta_3 \neq 0$, $\xi_3/\eta_3 \neq 1$. Also

$$(P_0 P_1 : P_2 P_3) = \frac{\xi_3}{\eta_3} = \lambda$$

Then we can write, in a suitable coordinate system,

$P_0 = [1:0]$, $P_1 = [0:1]$, $P_2 = [1:1]$, $P_3 = [\lambda : 1]$ with $\lambda \in \mathbb{F} \setminus \{0,1\}$

and then $(P_0 P_1 : P_2 P_3) = \lambda$.

Same for Q's, in a different coordinate system. $\exists$ unique τ which takes $P_0, P_1, P_2 \to Q_0, Q_1, Q_2$. Then $P_3 \longmapsto Q_3$ if and only if $(P_0 P_1 : P_2 P_3) = (Q_0 Q_1 : Q_2 Q_3)$

$\square$