Summary

$$\left\{ \begin{matrix} \text{ordered triples of distinct} \\ (P_1, P_2, P_3) \in \mathbb{F}\mathbb{P}^1 \end{matrix} \right\} \Big/ PGL(\mathbb{F}^2) \quad \xleftrightarrow{1-1} \quad \{ * \}$$

$$\left\{ \begin{matrix} \text{ordered quadruples of distinct} \\ \text{points } (P_0, P_1, P_2, P_3) \in \mathbb{F}\mathbb{P}^1 \end{matrix} \right\} \Big/ PGL(\mathbb{F}^2) \quad \longleftrightarrow \quad \mathbb{F} \setminus \{0, 1\}$$

$$(P_0, P_1, P_2, P_3) \longrightarrow (P_0 P_1 : P_2 P_3) = \lambda \in \mathbb{F} \setminus \{0,1\}$$

[Larger sets $\rightsquigarrow$ interesting geometry!]

In coordinates $(P_0, P_1, P_2) \xrightarrow[PGL(\mathbb{F}^2)]{} ([0:1], [1:0], [1:1])$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \qquad \infty \qquad 1$

$(P_0, P_1, P_2, P_3) \xrightarrow[PGL(\mathbb{F}^2)]{} ([0:1], [1:0], [1:1], [\lambda:1])$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \qquad \infty \qquad 1 \qquad \lambda$

Abstract projective geometry

Abstract projective plane Π is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is the set of points of Π, $\mathcal{L}$ is the set of lines, and $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is a relation: for $p \in \mathcal{P}$ point, $L \in \mathcal{L}$ line, $(p, L) \in \mathcal{I}$ means that point p is on line L.

This setup should be subject to the following <u>axioms</u>:

i, $\forall$ $p, q \in \mathcal{P}$, $p \neq q$, $\exists!$ (unique) $L \in \mathcal{L}$ such that $(p, L) \in I$ and $(q, L) \in I$

Through any two distinct points, there passes a unique line.

ii, Any two lines meet in at least one point.

iii, Any line contains at least three [redacted] distinct points

iv, There exist two distinct lines.

Example: given a field $\mathbb{F}$, then $\mathbb{F}\mathbb{P}^2 = \mathbb{P}(\mathbb{F}^3)$ satisfies these axioms.

eg: iii, $L = \mathbb{P}(U)$, U has basis $\{\underline{u}_1, \underline{u}_2\}$; take $[\underline{u}_1]$, $[\underline{u}_2]$, $[\underline{u}_1 + \underline{u}_2]$ are distinct points on L

In coordinates, this would be $(1:0:0)$, $(0:1:0)$ and $(1:1:0)$.

Question: can we go back? Given an abstract projective plane $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ such that $(\mathcal{P}, \mathcal{L}, \mathcal{I}) \cong \mathbb{F}\mathbb{P}^2$?

Answer: in general, no! But...

Assumption 1 / Further axiom D: Let us further require that Desargues' theorem holds in $(\mathcal{P}, \mathcal{L}, \mathcal{I})$.

Note: this is indeed an extra axiom which does not follow from previous axioms.

<u>Recall</u>: $\mathbb{FP}^2 = \mathbb{F}^2 \amalg \{\text{line at } \infty\}$

<u>Start with</u> $\mathcal{P} = L \amalg \mathcal{A}$, fix also any point $O \in \mathcal{A}$.

$\uparrow$

point set of L

i.e. points contained in L

(choose any line in $\mathcal{L}$)

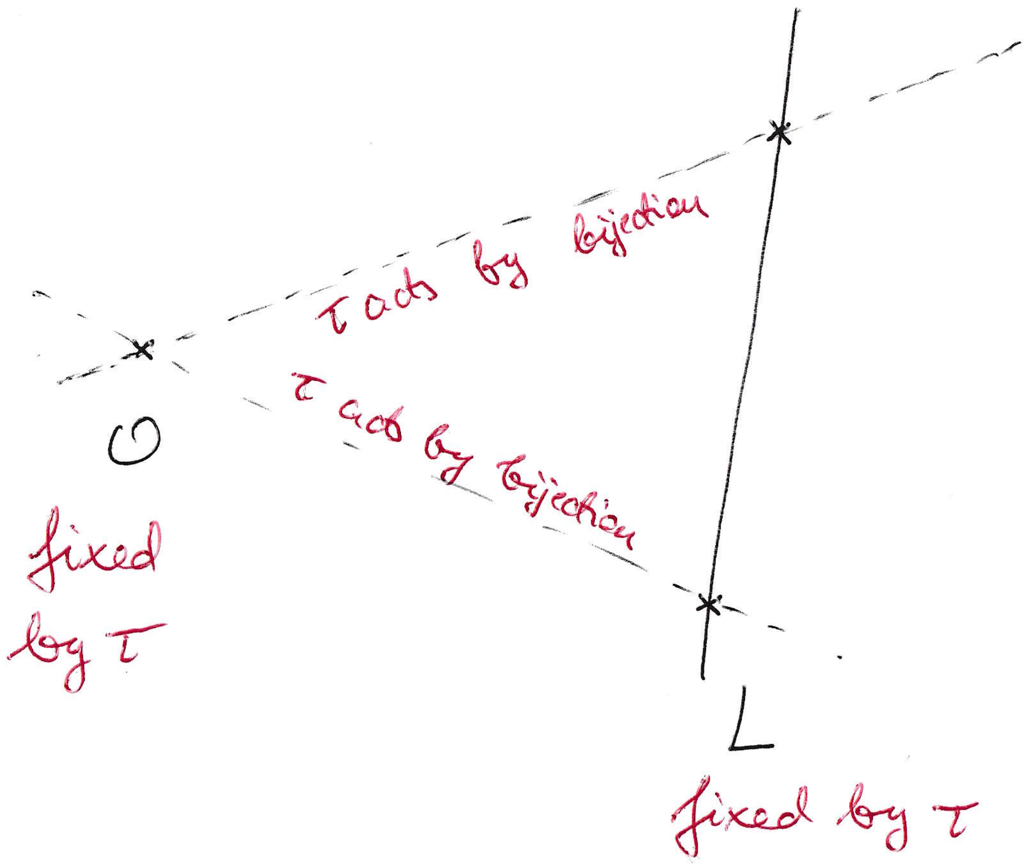
<u>Def 1</u> A <u>collineation</u> is bijections $\left\{\begin{matrix} \mathcal{P} \to \mathcal{P} \\ \mathcal{L} \to \mathcal{L} \end{matrix}\right\}$ preserving incidence relations.

<u>Def 2</u> A <u>central collineation</u> for (L, O) is a collineation $\tau : \begin{matrix} \mathcal{P} \to \mathcal{P} \\ \mathcal{L} \to \mathcal{L} \end{matrix}$

such that a) $\tau(O) = O$

b) τ fixes each point on L

$\Longrightarrow$ lines through O are mapped to themselves (not pointwise)



In $\mathbb{FP}^2$ these would just be scalings; use coordinates $[x_0 : x_1 : x_2]$ then $O = [0:0:1]$, $\{x_2 = 0\} = L$ are suitable choices, and examples of central collineations are $[x_0 : x_1 : x_2] \mapsto [\lambda x_0 : \lambda x_1 : x_2]$ for $\lambda \in \mathbb{F}^*$ (and in fact these are all such)

Let $K = \left\{ \tau : \begin{matrix} \mathcal{L} \to \mathcal{L} \\ \mathcal{P} \to \mathcal{P} \end{matrix}\right.$ central collineations with respect to $(L, O) \big\} \cup \{0\}$

"zero"

This set K carries "multiplication" defined by composition

$$\tau_1 \cdot \tau_2 = \begin{cases} \tau_1 \circ \tau_2 & \text{if } \tau_1, \tau_2 \text{ central collineations} \\ 0 & \text{if } \tau_1 = 0 \text{ or } \tau_2 = 0 \end{cases}$$

K also carries <u>addition</u> operation which can be defined geometrically; from its definition it is clear that it is commutative.

<u>Note finally</u> that K acts on $\mathcal{A}$: $\tau \in K$ has $\tau : L \to L$ (pointwise) so $\tau|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, or $\tau = 0 \in K$ acts by mapping everything to $O \in \mathcal{A}$.

Theorem a.) $(K, +, \cdot)$ is associative and distributive, and has division, with respect to $1 =$ identity collineation (K is a skew field)

b.)
$$\begin{tikzcd} \mathcal{A} \arrow[r, leftrightarrow] \arrow[loop below, "K"] & K^2 \arrow[loop below, "\text{multiplication action by } K"] \end{tikzcd}$$
and $\Pi \longleftrightarrow KP^2 = P(K^3)$

c.) $(K, \cdot)$ commutative $\Longleftrightarrow$ Pappus' Theorem holds in $(\mathcal{P}, \mathcal{L}, \mathcal{I}) = \Pi$

Comments - Associativity in a.) follows from Axiom D (Desargues' thm)

- c.) $\Longleftrightarrow K = \mathbb{F}$ field, and then $(\mathcal{P}, \mathcal{L}, \mathcal{I}) \cong \mathbb{F}P^2$.

Correction Pappus $\Longrightarrow$ Desargues (Hessenberg's theorem)

Duality - for projective plane

The entire axiomatics is completely symmetric: $\mathcal{P}$ points / $\mathcal{L}$ lines could interchange roles: $\begin{cases} \mathcal{L}: \text{set of points} \\ \mathcal{P}: \text{set of lines} \end{cases}$ and then axioms continue to hold!

i) For any distinct lines, there is a unique point incident to them

ii) For any distinct points there is a unique line - - - -

iii)-iv) "enough lines and points on them"

Start <u>dualizing</u> more complicated statements! $\mathcal{D}$ and $\mathcal{P}$ in particular.

<u>Example</u> $\mathcal{D}^*$ (dual of Desargues): let $\pi, \alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be lines in a projective plane, such that $\alpha \cap \alpha'$, $\beta \cap \beta'$, $\gamma \cap \gamma'$ are distinct and lie on line π. Then the lines $\overline{(\alpha \cap \beta)\,(\alpha' \cap \beta')}$, $\overline{(\alpha \cap \gamma)\,(\alpha' \cap \gamma')}$, $\overline{(\beta \cap \gamma)(\beta' \cap \gamma')}$ are all going through a common point.

<u>Principle of Duality</u>: $\mathcal{D} \Longrightarrow \mathcal{D}^*$ in any projective plane.

$\uparrow$

axiom

or in $\mathbb{F}P^2$, proved