Projective Geometry

Lecture 6

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Duality on projective planes

Recall **abstract projective plane** $\Pi = \{\mathcal{P}, \mathcal{L}, \mathcal{I}\}$

- $\mathcal{P}$ is the set of points;
- $\mathcal{L}$ is the set of lines;
- $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is the incidence relation between points and lines;
- subject to some axioms.

Duality: Following Hilbert, we can think instead of $\mathcal{L}$ is the set of points; $\mathcal{P}$ is the set of lines; axioms will remain the same!

This allows us to deduce new theorems from old, dualizing an earlier statement. Example: Desargues' Theorem.

To extend duality to higher dimensions, linear algebra will again be very helpful.

A little bit of Part A Linear Algebra

We recall that to any vector space V over a field $\mathbb{F}$ we can associate the **dual space**

$$V^* = \{f : V \to \mathbb{F} \text{ linear}\}.$$

If $\dim V = n$, then V and V^* are isomorphic, since V^* also has dimension n. However, this isomorphism depends on a choice of basis.

Recall that if $\{e_1, \ldots e_n\}$ is a basis of V, then a basis of V^* is given by the **dual basis** $\{E_1, \ldots, E_n\}$, defined by

$$E_i(e_j) = \delta_{ij}$$

and extended linearly.

The double dual V^{**}, that is, the dual of V^*, **is canonically** isomorphic to V. Explicitly, the map

$$\begin{aligned} \varphi : \ V &\to V^{**} \\ v &\mapsto (f \mapsto f(v) \text{ for } f \in V^*) \end{aligned}$$

defines an isomorphism between V and V^{**}.

A little bit more Part A Linear Algebra

Further, given a linear subspace $U \leq V$, we have its annihilator

$$U^\circ = \{f \in V^* : f(u) = 0 \text{ for all } u \in U\}.$$

Proposition For subspaces U, U_1, U_2 of a finite-dimensional vector space V

(i) if $U_1 \leq U_2$, then $U_2^\circ \leq U_1^\circ$; that is, taking the annihilator reverses inclusion;

(ii) $(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ$;

(iii) $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$;

(iv) $\dim U + \dim U^\circ = \dim V$;

(v) $(U^\circ)^\circ = \varphi(U)$.

Conclusion: get an inclusion-reversing one-to-one correspondence $U \leftrightarrow U^\circ$ between linear subspaces of V and linear subspaces of V^*.

Note: We shall use the canonical isomorphism φ to identify spaces with their double duals, and subspaces with their double annihilators, without further comment.

Duality of projective subspaces

Let now $\dim V = n+1$ and consider n-dimensional projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$.

We obtain an inclusion-reversing **duality correspondence**

$$\{\text{linear subspaces } \mathbb{P}(U) \subset \mathbb{P}(V)\} \longleftrightarrow \{\text{linear subspaces } \mathbb{P}(U^\circ) \subset \mathbb{P}(V^*)\}.$$

By the dimension formula, if $\mathbb{P}(U)$ is an m-dimensional linear subspace of $\mathbb{P}^n = \mathbb{P}(V)$, then...
...U has dimension $m+1$
...so U° has dimension $(n+1)-(m+1) = n-m$
...and hence $\mathbb{P}(U^\circ)$ is a linear subspace of $\mathbb{P}(V^*)$ of dimension $n-m-1$.

From the Proposition above, we also get

$$\langle \mathbb{P}(U_1), \mathbb{P}(U_2)\rangle^\circ = \mathbb{P}(U_1^\circ) \cap \mathbb{P}(U_2^\circ)$$

$$(\mathbb{P}(U_1) \cap \mathbb{P}(U_2))^\circ = \langle \mathbb{P}(U_1^\circ), \mathbb{P}(U_2^\circ)\rangle.$$

Duality of points and hyperplanes

Points of $\mathbb{P}(V^*)$ represent 1-dimensional subspaces of V^*. These correspond to hyperplanes in $\mathbb{P}(V)$, which represent n-dimensional subspaces of V.

The correspondence assigns to $\langle f \rangle$, where $f \in V^* - \{0\}$, the hyperplane $\mathbb{P}(\ker(f))$ in $\mathbb{P}(V)$.

In homogeneous coordinates, the point $[a_0 : \ldots : a_n]$ in the dual projective space $\mathbb{P}(V^*)$ corresponds to the hyperplane

$$\{a_0 x_0 + \ldots + a_n x_n = 0\} \subset \mathbb{P}(V).$$

Note that scaling all the a_i does not alter the hyperplane.

Conversely, hyperplanes in $\mathbb{P}(V^*)$ correspond to points in $\mathbb{P}(V^{**})$ and thus to points in $\mathbb{P}(V)$.

Duality of points and lines in the projective plane

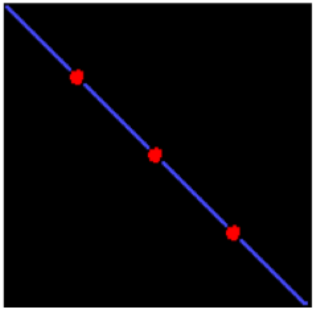
Assume $\dim V = 3$ so $\dim \mathbb{P}(V) = 2$.

Duality interchanges points of $\mathbb{P}(V) = \mathbb{FP}^2$ and lines in $\mathbb{P}(V^*) = \mathbb{FP}^2$.

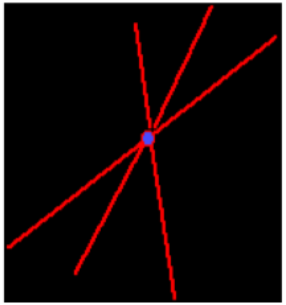
If $P = [p], Q = [q]$ are two distinct points on the line $L = \mathbb{P}U \subset \mathbb{P}(V)$ with $U = \langle p, q \rangle$, then the lines $\mathbb{P}\langle p \rangle^\circ, \mathbb{P}\langle q \rangle^\circ$ meet at the point $\mathbb{P}U^\circ$ of $\mathbb{P}(V^*)$.

More generally, a set of collinear points in $\mathbb{P}(V)$ corresponds under duality to a set of **concurrent** lines in $\mathbb{P}(V^*)$ (lines passing through a common point).

three collinear points



three concurrent lines



Points and lines in general position in the projective plane

On a projective plane $\mathbb{P}(V) = \mathbb{F}\mathbb{P}^2$ we can define four **lines** to be in general position if no three of them are concurrent.

This is equivalent to the four points they represent in $\mathbb{P}(V^*)$ being in general position.

Under duality, a line

$$\{\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0\} \subset \mathbb{P}(V)$$

corresponds to the point

$$[\alpha_0 \colon \alpha_1 \colon \alpha_2] \in \mathbb{P}(V^*).$$

So by the General Position Theorem, four lines in $\mathbb{P}(V)$ which are in general position can be assumed to have the equations

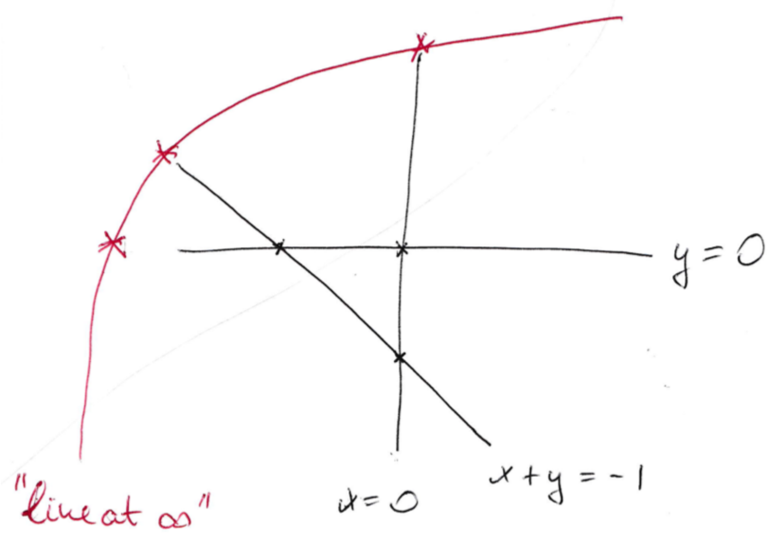
$$x_0 = 0, \qquad x_1 = 0, \qquad x_2 = 0, \qquad x_0 + x_1 + x_2 = 0.$$

Lines in general position in the projective plane

Four lines in $\mathbb{P}(V)$ which are in general position can be assumed to have the equations

$$x_0 = 0, \qquad x_1 = 0, \qquad x_2 = 0, \qquad x_0 + x_1 + x_2 = 0.$$

In affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, we get the following picture.



Real conics in the plane $\mathbb{R}^2$

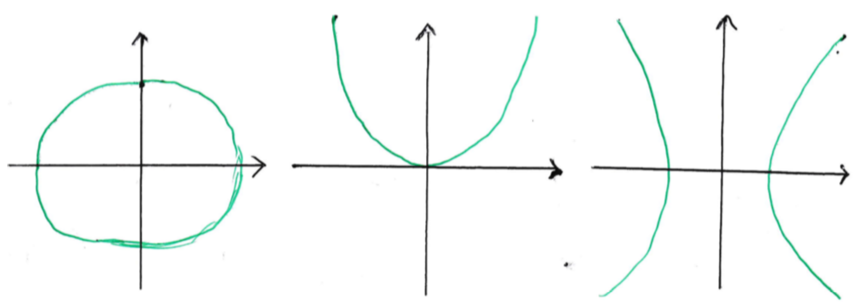
A **conic** is a plane curve given by a quadratic equation.

A real (affine) conic is a curve $C \subset \mathbb{R}^2$ given by a quadratic equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

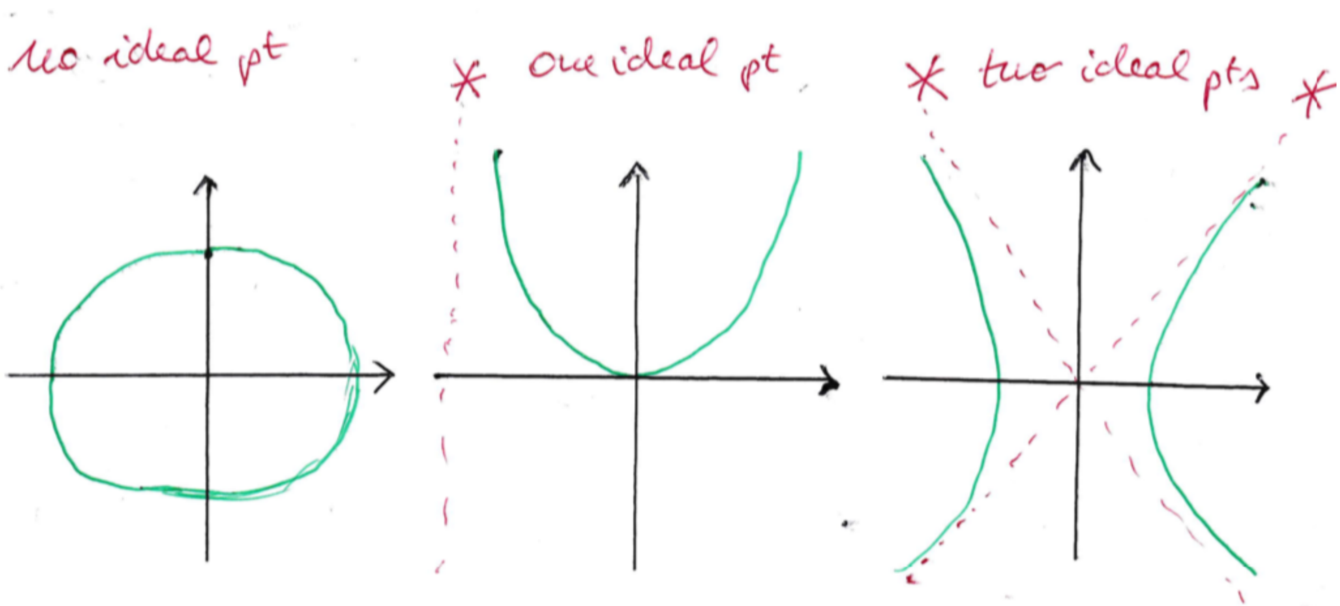
Three types of conics: **ellipse, parabola, hyperbola**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad\qquad y = ax^2 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Real conics in the plane $\mathbb{R}^2$ and their asymptotes

Where do these conics meet the **ideal line**?



Real conics: extending to the projective plane

Recall that the transformation between projective and affine coordinates is $x = x_1/x_0$, $y = y_1/y_0$.

We get the projective plane $\mathbb{RP}^2$ with coordinates $[x_0 : x_1 : x_2]$, its ideal line $L_\infty = [0 : x_1 : x_2]$ and the ordinary plane $\mathbb{R}^2 = [1 : x : y]$.

We get the projective equations

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = x_0^2 \qquad x_0 x_2 = a x_1^2 \qquad \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = x_0^2$$

The real projective solutions with $x_0 = 0$ (ideal points) are

$$\emptyset \qquad [0 : 0 : 1] \qquad [0 : a : \pm b]$$

Note that, after a linear change coordinates, all three equations have the form

$$-y_0^2 + y_1^2 + y_2^2 = 0.$$

Bilinear forms over arbitrary fields

A **symmetric bilinear form** on a vector space V over $\mathbb{F}$ is a map

$$B : V \times V \to \mathbb{F}$$

such that

(i) $B(v, w) = B(w, v)$;

(ii) B is linear in v (and hence, by (i), in w).

If an addition we have that

(iii) if $B(v, w) = 0$ for all w, then $v = 0$,

then we say the form is **nondegenerate** or **nonsingular**.

Remark Note that the conditions are different from those seen in other contexts. Over $\mathbb{F} = \mathbb{R}$, we could require positive definiteness instead of (iii). Over $\mathbb{F} = \mathbb{C}$, we could require sesquilinearity instead of (i)-(ii). Our conditions make sense for any field.

Bilinear forms over arbitrary fields

If we choose a basis $\{e_0, \ldots, e_n\}$ of V, then a bilinear form is given by

$$B(v, w) = v^t X w$$

for a symmetric matrix X given by

$$X_{ij} = B(e_i, e_j).$$

Nondegeneracy of the form is equivalent to nonsingularity (invertibility) of the matrix X.

A bilinear form is determined (if the characteristic of $\mathbb{F}$ is $\neq 2$), by the associated **quadratic form**

$$Q(v) = B(v, v),$$

for we can recover B via the polarisation identity

$$B(v, w) = \frac{1}{4}(B(v + w, v + w) - B(v - w, v - w)) = \frac{1}{4}(Q(v + w) - Q(v - w))$$

Projective quadrics and conics over arbitrary fields

A **projective quadric** is the locus of points in a projective space $\mathbb{P}(V)$ defined by an equation $Q(v) = 0$, where $v \mapsto Q(v) = B(v, v)$ is a (not identically zero) quadratic form on V.

A **projective conic** is a projective quadric in a projective plane $\mathbb{P}(V) = \mathbb{FP}^2$.

Projective transformations send quadrics to quadrics. If we write the quadratic form in terms of a symmetric matrix X, then its image under a projective transformation is the form defined by the symmetric matrix $M^t X M$, where M defines the projective transformation.

Note also that if Q and Q' are proportional, that is $Q'(v) = \lambda Q(v)$ for all v, then they define the same quadric.

Definition We say a quadric is **nonsingular**, if the associated symmetric bilinear form is nondegenerate.