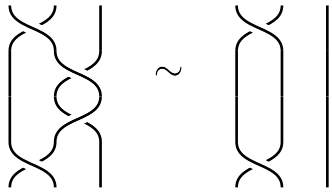
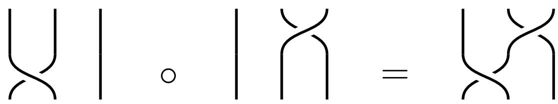
The Braid Group

Introduction

The braid group Br_n is the group whose elements are equivalence classes of n-braids. The braid group is important in many branches of mathematics, and shows up in physics too. An n-braid is an arrangement of n strands connecting n fixed points to n fixed points, moving in one direction (so not going backward). The equivalence relation on n-braids given by *isotopy*, or "pulling the strands":



We compose two n-braids by gluing the endpoints of the first to the starting points of the second:



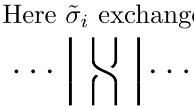
The inverse of a braid is given by "doing the braid backwards".

Presentation of the Braid Group

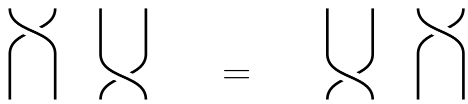
Elements in the braid group can be generated by single exchanges of two adjacent strands. This renders the following presentation of the braid group:

$$Br_n = \left\langle \tilde{\sigma}_1, ..., \tilde{\sigma}_{n-1} \;\middle|\; \begin{array}{ll} \tilde{\sigma}_i\tilde{\sigma}_j = \tilde{\sigma}_j\tilde{\sigma}_i & \text{for } |i-j| \geq 2; \\ \tilde{\sigma}_i\tilde{\sigma}_{i+1}\tilde{\sigma}_i = \tilde{\sigma}_{i+1}\tilde{\sigma}_i\tilde{\sigma}_{i+1} & \text{for } 1 \leq i \leq n-2 \end{array} \right\rangle \tag{1}$$

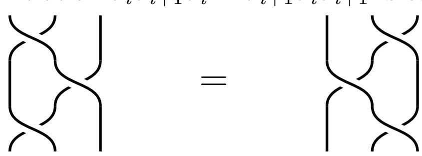
Here $\tilde{\sigma}_i$ exchanges strands i and $i+1$ and leaves the rest unchanged:



The relation $\tilde{\sigma}_i\tilde{\sigma}_j = \tilde{\sigma}_j\tilde{\sigma}_i$ for $|i-j| \geq 2$ states that generators involving disjoint sets of strands commute:



The relation $\tilde{\sigma}_i\tilde{\sigma}_{i+1}\tilde{\sigma}_i = \tilde{\sigma}_{i+1}\tilde{\sigma}_i\tilde{\sigma}_{i+1}$ is called the braid relation:

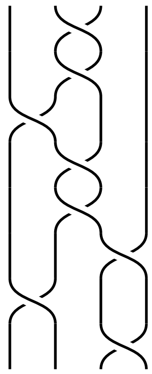


The Pure Braid Group

Let S_n be the symmetric group on n letters, which admits the following presentation:

$$S_n = \left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i^2 = e & \text{for all } i; \\ \sigma_j \sigma_i & \text{for } |i-j| \geq 2; \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } 1 \leq i \leq n-2 \end{array} \right\rangle \qquad (2)$$

Here σ_i is the transposition $(i \;\; i+1)$. There is a homomorphism $\phi : B_n \to S_n$ taking $\tilde{\sigma}_i \mapsto \sigma_i$. It is easy to see that this is a homomorphism: the relations in the presentation (1) of Br_n are mapped to relations in the presentation (2) of S_n. The kernel of this homomorphism is the pure braid group PBr_n, whose elements are equivalence classes of braids in which the strands return to the same position, for example:



Homomorphism to $\mathbb{Z}$

Define $\phi : Br_n \to \mathbb{Z}$ by $\tilde{\sigma}_i \mapsto 1$ for each i. We want to show that this map defines a homomorphism.

$$\begin{array}{rcccccc} \phi(\tilde{\sigma}_i \tilde{\sigma}_j) & = & \phi(\tilde{\sigma}_i)\phi(\tilde{\sigma}_j) & = & 2 & = & \phi(\tilde{\sigma}_j \tilde{\sigma}_i) \\ & & \phi(\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i) & = & 3 & = & \phi(\tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1}) \end{array}$$

Thus the relations in the presentation of Br_n are respected in the image of ϕ, hence ϕ defines a homomorphism. By the first isomorphism theorem,

$$Br_n / \ker \phi \cong \operatorname{im} \phi \cong \mathbb{Z}.$$

This shows that each $\tilde{\sigma}_i$ has infinite order as the composite $\langle \tilde{\sigma}_i \rangle \to Br_n \to \mathbb{Z}$ is an isomorphism.

Abelinization of the Braid Group

In the abelinization $(Br_n)_{ab}$ of the braid group, the relations $\tilde{\sigma}_i \tilde{\sigma}_j = \tilde{\sigma}_j \tilde{\sigma}_i$ are added for all i, j. The braid relation then states:

$$\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i^2 = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1} = \tilde{\sigma}_{i+1}^2 \tilde{\sigma}_i,$$

hence $\tilde{\sigma}_{i+1} = \tilde{\sigma}_i$ for all i. Therefore $(Br_n)_{ab} \cong \langle \tilde{\sigma}_1 \rangle \cong \mathbb{Z}$. Then the derived group of the braid group is $Br_n' \cong \ker \phi$, for ϕ as in paragraph 1.4.