4. Composition series and the Jordan-Hölder theorem

Def: A sequence of subgroups

$$G_0 = \{e\} \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

such that G_{i-1} is normal in G_i is called a composition series if each composition factor G_{i+1}/G_i is simple.

Prop: Let G be a non-trivial and finite group. Then G has a composition series.

proof: Proceed by induction on the size of the group, $|G|$.
If G is simple we are done: $G_0 = \{e\} \lhd G_1 = G$
Otherwise G has a non-trivial normal subgroup N with:
$|G/N| < |G|$ and $|N| < |G|$, and hence, by induction, both have composition series:
$\{e\} = Q_0 \lhd Q_1 \lhd \cdots \lhd G/N = Q_k$ and $0 \lhd S_1 \cdots S_{m-1} \lhd N = S_m$
Put $S_{i+m} := \pi^{-1}(Q_i)$

Then

$$\{e\} \lhd S_1 \cdots S_m \lhd S_{m+1} \lhd \cdots \lhd G$$

$S_{i+1}/S_i = Q_{i+1}/Q_i$ $(i \geq m)$

is a composition series by the Third Isom. Th. □

Ex: $C_{12} \rhd C_6 \rhd C_3 \rhd \{e\}$ $\quad C_{12}/C_6 \cong C_2,\ C_6/C_3 \cong C_2,\ C_3$
$C_{12} \rhd C_4 \rhd C_2 \rhd \{e\}$ $\quad C_{12}/C_4 \cong C_3,\ C_4/C_2 \cong C_2,\ C_2$

$S_4 \rhd A_4 \rhd V_4 \rhd C_2 \rhd \{e\}$

Note: If G is finite abelian then all composition factors are cyclic of prime order.

Th (Jordan - Hölder theorem)

Let G be a finite, non-trivial group.
Then all composition series have the same length and, up to reordering, the same composition factors with multiplicities.

proof: Assume we have composition series

$$G = G_r \triangleright \cdots \triangleright G_0 = \{e\}$$
$$G = H_s \triangleright \cdots \triangleright H_0 = \{e\}$$

Proceed by induction on r with $r \geq s$.

If $r = 1$, $G = G_1$ is simple has only one comp. series.

Assume $r > 1$ and the theorem holds for any group having a comp series of length $r-1$

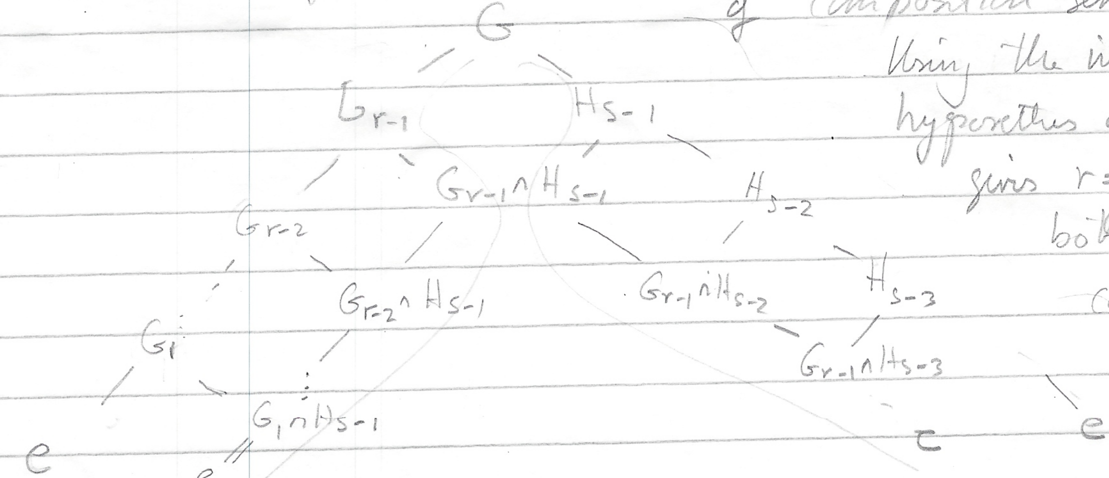
If $G_{r-1} = H_{s-1}$, then by assumption the comp. series for $G_{r-1} = H_{s-1}$ satisfy the theorem, and hence for $G_r = H_s$ as $G_r / G_{r-1} \cong H_s / H_{s-1}$ and $r = s$.

If $G_{r-1} \neq H_{s-1}$, consider

$$G \triangleright G_{r-1} \triangleright G_{r-1} \cap H_{s-1} \triangleright \quad \cdots \quad \triangleright G_0 = \{e\}$$
$$G \triangleright H_{s-1} \triangleright \underset{\overset{\shortparallel}{K}}{G_{r-1} \cap H_{s-1}} \triangleright \quad \cdots \quad \triangleright H_0 = \{e\}$$

Note that $G = G_{r-1} H_{s-1}$ as $G_{r-1} H_{s-1} \triangleright_{\neq} G_{r-1}$.

Repeated use of the Diamond Lemma gives an interlocking diagram of composition series.



Using the induction hypothesis on $G_{r-1} \cap H_{s-1}$ gives $r = s$ and both have the same composition factors

5. Solvable groups

Def: For a group G and elements $x, y \in G$, define the commutator as

$$[x,y] = xyx^{-1}y^{-1}$$

and the derived group of G as

$$G' = \langle [x,y] \mid x, y \in G \rangle$$

Th: (1) G' is a normal subgroup of G. Furthermore, $G^{(n)}$ is normal in G.

(2) For $N \trianglelefteq G$, G/N is abelian iff $N \geq G'$

$G_{ab} = G/G'$ is the largest abelian quotient, the abelianization of G.

proof: (1) For any automorphism $\phi : G \to G$,

$\phi(G') = G'$ as $\phi([x,y]) = \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1}$
$= [\phi(x), \phi(y)]$

G' is a characteristic subgroup of G

For $g \in G$, $c_g(x) := g^{-1}xg$ is an automorphism and hence $g^{-1}[x,y]g = [g^{-1}xg, g^{-1}yg] \in G'$.

(2) For $g_1, g_2 \in G$, $(g_1N)(g_2N) = (g_2N)(g_1N)$
iff $g_1g_2N = g_2g_1N$ iff $[g_1, g_2] \in N$

So G/N is abelian iff $N \geq G'$. □

Def: Let $G^{(1)} = G'$ and $G^{(n+1)} = (G^{(n)})'$. Then the derived series of G is

$$\underbrace{G \trianglerighteq G' \trianglerighteq G'' \trianglerighteq}_{\trianglerighteq} \cdots \trianglerighteq G^{(n)} \trianglerighteq \cdots$$

G is solvable (or soluble) if $G^{(n)} = e$ for some n.
The first such n is called the derived length.

Prop: $G^{(n)}$ is normal in G for all $n \geq 1$.

proof: $G^{(n)} = (G^{(n-1)})'$ is characteristic in $G^{(n-1)}$.
By induction, $G^{(n-1)} \trianglelefteq G$ and c_g is an automorphism $G^{(n-1)} \to G^{(n-1)}$ for all $g \in G$.

Ex : $G = S_4$ with normal subgroups $A_4, V_4, \{e\}$;
$G' = A_4$ as $S_4/A_4 \cong C_2$ is abelian but S_4/V_4, $S_4/\{e\}$ are not
$G'' = (A_4)' = V_4$
as $A_4/V_4 \cong C_3$ is abelian but $A_4/\{e\}$ is not
$G^{(3)} = (V_4)' = \{e\}$
as V_4 is abelian
So S_4 is solvable with derived series

$$\underbrace{S_4 \trianglerighteq A_4 \trianglerighteq V_4}_{\triangleright} \trianglerighteq e \qquad \text{of length } 3$$

Ex : A_5 is simple and not abelian. So $A_5' = A_5$.
so A_5 is not solvable: $S_5 \trianglerighteq A_5 \trianglerighteq A_5 \trianglerighteq A_5 \ldots$

8a)

Th: Let G be a finite group.
Then G is solvable if and only if every composition factor is abelian (i.e. cyclic of prime order).

proof: "$\Rightarrow$" If G is solvable of length n then

$$(*) \qquad G \trianglerighteq G' \trianglerighteq G'' \trianglerighteq \cdots G^{(n-1)} \trianglerighteq G^{(n)} = \{e\}$$

with $G^{(n-1)}/G^{(n)}$ abelian.
Each $G^{(n-1)}/G^{(n)}$ has composition series with composition factors all abelian, and $(*)$ can be refined to a composition series with all abelian factors.

Using the subgroup correspondence

"$\Leftarrow$" Let $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_r = \{e\}$ be a composition series with G_i/G_{i-1} abelian.
Proceed by induction on k: to prove $G_k \geq G^{(k)}$ for $k = 1, \ldots, r$
G_0/G_1 is abelian, and so $G_1 \geq G'$ (by Th, (3))
Assume $G_k \geq G^{(k)}$. Then $G_k' \geq (G^{(k)})' = G^{(k+1)}$.
G_k/G_{k+1} is abelian, and so $G_{k+1} \geq G_k' \geq G^{(k+1)}$.
In particular, $G_r = \{e\} = G^{(r)}$ and G is solvable.

$G/G' = G_{ab}$ is the abelinization of G, i.e. the largest abelian quotient of G

If $G = \langle X / R \rangle = F(X) / \langle\langle R \rangle\rangle$
then

$$G_{ab} = \langle X \mid R \cup \{ [x,y] \mid x,y \in X \} \rangle$$

$\underline{Ex_1}$: $Br_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| = 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \end{array} \right\rangle$

$(Br_n)_{ab}$: add relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for all i, j
hence, the braid relation gives
$\sigma_i = \sigma_{i+1}$ for all i

$$\Rightarrow (Br_n)_{ab} \simeq \langle \sigma_1 \rangle \simeq \mathbb{Z}$$

and $Br_n' = \ker \left(\begin{array}{c} \phi : Br_n \to \mathbb{Z} \\ \sigma_i \mapsto 1 \end{array} \right)$

$\underline{Ex_2}$: $F_n = \langle a_1, \dots, a_n \mid \phi \rangle$ free group on n-generators

$$(F_n)_{ab} = \langle a_1, \dots, a_n \mid [a_i, a_j] \rangle \simeq \mathbb{Z}^n$$

$\underline{Ex_3}$: If G is abelian, then $G = G_{ab}$ and its derived length is 1.

6. Semi-direct Products

<u>Def</u>: Let G be a group, $H \leq G$, $N \trianglelefteq G$.
Then G is an <u>internal semi-direct product</u> of H and N if $G = NH$ and $N \cap H = \{e\}$.
Write
$$G = N \rtimes H \qquad \text{(equivalently, } G = H \ltimes N\text{)}$$

<u>Note</u>: In this case $G/N \cong H$.

<u>Ex</u>: Direct-product $G = H \times N$.

<u>Ex</u>:
$$D_{2n} = C_n \rtimes C_2$$
$$S_n = A_n \rtimes C_2$$
$$S_4 = V_4 \rtimes S_3$$

<u>Recall</u>: $n_1, n_2 \in N$, $h_1, h_2 \in H$,
$$(n_1 h_1)(n_2 h_2) = n_1 (h_1 n_2 h_1^{-1}) h_1 h_2$$
multiplication is twisted by conjugation by h_1
$$= n_1 \, c_{h_1}(n_2) \, h_1 h_2$$
$c_h(x) = h x h^{-1}$ $\quad c_h \in \mathrm{Aut}(G)$

<u>Lemma</u>: $N \trianglelefteq G$, $H \leq G$.
Then $H \to \mathrm{Aut}(N)$ via $h \mapsto c_h$ is a homom.

<u>proof</u>: We have already seen that
$$G \xrightarrow{c} \mathrm{Aut}(G) \qquad g \longmapsto c_g$$
is a homomorphism.
Hence, the restriction to $H \leq G$ is a homomorphism.
As N is normal in G, for all $g \in G$ (hence all $h \in H$)
$$c_g(n) = g n g^{-1} \in N \text{ for all } n \in N.$$
So c_g restricts to an element in $\mathrm{Aut}(N)$.

Def/Th: given two groups H and N, and a homomorphism
$$\phi : H \to \mathrm{Aut}(N)$$
define a group, the semi-direct product $N \rtimes_\phi H$ with elements (n,h) $n \in N$, $h \in H$ and product
$$(n_1,h_1)(n_2,h_2) = (n_1\,\phi(h_1)(n_2),\ h_1 h_2).$$
Note $N \times \{e\} \trianglelefteq N \rtimes_\phi H$ and $\{e\} \times H \leq N \rtimes_\phi H$.

Ex: $N = C_n$
$\mathrm{Aut}(C_n) \simeq C_n^\times$ group of units in $C_n \cong \mathbb{Z}/n$
$= \{ k \mid (n,k) = 1 \}$
$\underline{n=3}$: $\mathrm{Aut}(C_3) \simeq \{1,2\} \simeq C_2$
$\underline{n=12}$: $\mathrm{Aut}(C_{12}) \simeq \{1,5,7,11\} \simeq C_2 \times C_2$

Ex: $N = C_3 = \langle x \rangle$ $H = C_4 = \langle y \rangle$
How many semi-direct products $C_3 \rtimes C_4$
$\phi : C_4 \to \mathrm{Aut}(C_3) = C_2 = \{1, \tau\}$, $\tau(x) = x^2 = x^{-1}$
ϕ is determined by $\phi(y)$
$\phi(y) = 1$ then $C_3 \rtimes_\phi C_4 = C_3 \times C_4$

$\phi(y) = \tau$ then $C_3 \rtimes C_4 = \langle x, y \mid x^3 = y^4 = e,\ yx = x^{-1} y \rangle$

Prop: H is normal in $G = N \rtimes_\phi H$ iff ϕ is trivial i.e. $G = N \times H$

proof: $n^{-1} h n \in H \iff n^{-1} \phi(h)(n)\, h \in H \iff n^{-1}\phi(h)(n) \in H$
$\iff \phi(h)(n) = n$ as $N \cap H = \{e\}$

$$\begin{array}{ccccc} N & & & & Q \\ \cong & & & & \cong \\ K & \hookrightarrow & G & \twoheadrightarrow & G/K \end{array}$$

<u>short exact sequence</u>

<u>Def</u>: Let N and Q be two groups.
An <u>extention</u> of N by Q is a group G such that G has a normal subgroup K such that

$$K \cong N \quad \text{and} \quad G/K \cong Q.$$

The extension is <u>split</u> if there is a subgroup $H \cong Q$ of G such that

$$\begin{tikzcd} H \arrow[r, hook] & G \arrow[r, two heads] & G/K \cong Q \arrow[ll, bend left, "s"] \end{tikzcd}$$

is an isomorphism.

<u>Prop</u>: G is a split extension of N by Q iff there exists $\phi : Q \to Aut(N)$ such that

$$G \cong N \rtimes_\phi Q.$$

<u>Ex</u>: C_6 is an extension of C_2 by C_3; it is split
C_8 is an extension of C_2 by C_4; it is <u>not</u> split
S_n is an extension of A_n by C_2; it is split.

<u>proof</u>: If $G = N \rtimes_\phi Q$ then $N \hookrightarrow G \overset{\pi}{\twoheadrightarrow} Q$ is split with $s : Q \to G$ the inclusion of the subgroups.
If $N \hookrightarrow G \twoheadrightarrow Q$ is split, we may identify Q via s with a subgroup of G.
Then $\phi : Q \to Aut(N)$ via $q \mapsto c_q : N \to N$, $n \mapsto q n q^{-1}$

defines a homomorphism and $N \rtimes_\phi Q$ is well-defined.
Consider $N \rtimes_\phi Q \xrightarrow{\alpha} G$ via $\alpha(n, q) = nq$.

- α is a homomorphism:

$$\alpha((n_1,q_1)(n_2,q_2)) = \alpha(n_1\, q_1 n_2 q_1^{-1},\, q_1 q_2) = n_1 q_1 n_2 q_1^{-1} q_1 q_2$$
$$= n_1 q_1 n_2 q_2 = \alpha(n_1,q_1)\,\alpha(n_2,q_2)$$

- α is injective: $\alpha(n,q) = e \iff n = q^{-1} \in N \cap Q$
$\iff n = q^{-1} = e$ as s is a splitting and π is injective on $s(Q)$.
- α is surjective: every $g \in G$ is contained in some $g \ker\pi$; but $G/\ker\pi \cong Q \leq G$ are the coset representatives.