7 Sylow Theorems

Recall : Lagrange : $H \leq G \Rightarrow |H| \,/\, |G|$
converse is not true : no subgroup of order 6 in A_4
partial converse:
Cauchy : $p \mid |G| \Rightarrow \exists$ subgroup $C_p \leq G$

Let G be finite with $|G| = p^\alpha m$ and $(p,m)=1$

Def: G is a p-group for a prime p if $|G| = p^\alpha$

Def: A subgroup P of G with $|P| = p^\alpha$ is a Sylow p-subgroup.
We let n_p be the number of different Sylow p-subgroups.

Sylow Theorems: $|G| = p^\alpha m$, $p \nmid m$
(1) G has a Sylow p-subgroup (i.e. $n_p \geq 1$)
(2) Any two Sylow p-subgroups P_1 and P_2 are conjugate (i.e. $\exists g \in G : g^{-1}P_1 g = P_2$)
(3) $n_p \equiv 1 \pmod p$ and $n_p \mid m$

Ex: $G = A_4$ $\quad |G| = 12 = 3 \cdot 2^2$
$p=3$: $\langle(123)\rangle, \langle(134)\rangle, \langle(124)\rangle, \langle(234)\rangle \sim C_3$ $\quad n_3 = 4$
$p=2$: V_4 $\quad V_4 \trianglelefteq A_4 \Rightarrow n_2 = 1$

$\beta \geq 1$ by Cauchy

Proof of the Sylow Theorems:

Let P be a maximal p-subgroup of G with $|P| = p^\beta$, $\beta \leq \alpha$.
Let $N_G P = \{ g \in G \mid g^{-1} N g = N \}$ be the normaliser of P in G; $P \trianglelefteq N_G P$

(**) $$p^\alpha m = |G| = |G : N_G P| \, |N_G P| = \underbrace{|G : N_G P|}_{n_p} \, |N_G P : P| \, |P|$$

Strategy for 1st Sylow Th: show $p \nmid |N_G P : P|$ and $p \nmid |G : N_G P|$
$\Rightarrow$ P is a Sylow p-subgroup as $|P| = p^\alpha$.

Step 1: Assume $p \mid |N_G P : P|$.
$\Rightarrow \exists A \leq N_G P / P$ with $|A| = p$ by Cauchy's Th
$\Rightarrow \exists \tilde{A} \leq N_G P$ with $|\tilde{A}| = p \cdot |P|$ by Subg. Corresp
This contradicts the maximality of P as $\tilde{A} \gneq P$.
So $p \nmid |N_G P : P|$

Step 2: Let Q be a p-subgroup of $N_G P$.
$\Rightarrow QP \leq N_G P$
$\Rightarrow QP/P \leq N_G P / P$
Assume $Q \nleq P$.
$\Rightarrow QP/P \cong P/Q \cap P$ by 2nd Iso Th
$\Rightarrow p \mid |N_G P : P|$ ↯ by Step 1
So $Q \leq P$.

Step 3: Let $\Sigma = \{ g(N_G P) \mid g \in G \} = G/N_G P$ and let P act on the left.
Define $\mathrm{Fix}_P(\Sigma) = \{ \alpha \in \Sigma \mid p\alpha = \alpha \text{ for all } p \in P \}$ — Fixed Point Set
$(\alpha = g(N_G P) \in \mathrm{Fix}_P \Sigma) \Leftrightarrow (\forall p \in P : pgN_G P = gN_G P \Leftrightarrow g^{-1}pg \in N_G P) \overset{\text{by step 2}}{\Leftrightarrow} g \in N_G P$
$\Leftrightarrow gN_G P = N_G P$

$$|\Sigma| = \bigoplus_{\text{orbits}} |\text{orbit}_\omega| = \bigoplus |P/\text{Stab}_\omega| \equiv \mathrm{Fix}_P \Sigma = 1 \mod p$$

used later

So $p \nmid |\Sigma| = |G : N_G P|$.

This proves 1st Sylow Theorem.

2nd Sylow Th follows from:

Claim: If Q is a p-subgroup of G then $\exists g: g^{-1}Qg \leq P$.

proof: Let Q act on $\Sigma = G/N_G P$.
As in Step 3
$$|\Sigma| \equiv |Fix_Q(\Sigma)| \mod p$$
As $p \nmid |\Sigma|$, $Fix_Q(\Sigma) \neq \emptyset$; say $gN_G P \in Fix_Q(\Sigma)$.
Then $\forall q \in Q$: $qgN_G P = gN_G P$, and hence
$$g^{-1}Qg \leq N_G P.$$
By Step 2, $g^{-1}Qg \leq P$. $\square$

3rd Sylow Th follows as:
$$\begin{aligned} n_p &= \#\text{ of conjugates of a Sylow } p\text{-subgroup } P \quad \text{— by 2nd Sylow Th} \\ &= |\{g^{-1}Pg \mid g \in G\}| \quad \longleftarrow G \text{ acts transitively via conj.} \\ &= |G| / \mathrm{Stab}_G(P) \\ &= |G| / |N_G(P)| \end{aligned}$$

and hence $n_p \equiv 1 \pmod p$ by the proof of Step 3
and $n_p \mid |G:P| = m$ by (**)
$$|G:P| = |G:N_G P|\,|N_G P:P|$$

Revision from Prelims: Group Actions

To say that a group G acts on a set S via

$$\phi : G \times S \to S$$
$$(g, s) \longmapsto \phi(g,s) = gs$$

is equivalent to say that

$$\phi : G \longrightarrow Sym(S)$$
$$g \longmapsto \phi(g, -)$$

is a group homomorphism.

Stabilizer - Orbit formula:

$$Orb(s) = \{ gs \mid g \in G \} \subseteq S$$
$$Stab(s) = \{ g \mid gs = s \} \subseteq G$$

(1) orbits partition S

(2) stabilizers are subgroups of G

(3) if G is finite then for all $s \in S$

$$|G| = |Stab(s)|\,|Orb(s)|$$

Counting formula:

$$Fix_G(S) = \{ s \in S \mid gs = s \ \forall g \in G \} = \{ s \in S \mid |Orb(s)| = 1 \}$$

Let $|G| = p^\alpha$ for a prime p.

Then

$$|Fix_G(S)| \equiv |S| \pmod p$$

Ex$_1$: G acts on G/H via $\phi(g, xH) = gxH$ for H a subgroup

Ex$_2$: G acts on G via $\phi(g, x) = gxg^{-1}$.

7.1 Classification of groups with small order

Recall:

(1) If p is a prime and $|G| = p$ then $G \simeq C_p$
(2) If $p \geq 3$ is prime and $|G| = 2p$ then $G \simeq C_{2p}$ or $G \simeq D_{2p}$
(3) If p is prime and $|G| = p^2$ then $G \simeq C_{p^2}$ or $G \simeq C_p \times C_p$

Prop: Let $p > q$ be two primes and $|G| = pq$.

(4) If $q \nmid p-1$ then $G \simeq C_{pq}$
(5) If $q \mid p-1$ then $G \simeq C_{pq}$ or $G \simeq C_p \rtimes_\phi C_q$ where $1 \neq \phi : C_q \to C_p^\times = \mathrm{Aut}(C_p)$

proof: (4+5) Let P and Q be Sylow p- and q- subgroups of G;
$n_p \equiv 1 \pmod p$ and $n_p \mid q$;
hence $n_p = 1$ and P is normal in G;
$\Rightarrow$ $G = PQ$ and G is a semi-direct product.
Consider possibilities for $\phi : Q \to \mathrm{Aut}\, P \simeq C_p^\times$
If $q \nmid p-1$ then ϕ is trivial and
$$G \simeq C_p \times C_q$$
If $q \mid p-1$ then ϕ could be trivial or is an injection to the unique cyclic subgroup of order q;

Prop:
(6) If $|G| = 8 = 2^3$ then $G \simeq C_8, C_4 \times C_2, C_2 \times C_2 \times C_2, D_8, Q_8 = \{\pm 1, i, j, k\}$
(7) If $|G| = 12 = 3 \cdot 2^2$ then $G \simeq C_3 \times C_4, C_3 \times C_2 \times C_2, D_6 \times C_2, A_4, C_3 \rtimes C_4$

Problem: Determine the non-abelian groups of order 12 upto isom.

Solution: Let G be a group with $|G| = 12$
P its Sylow 2-subgroup $|P| = 4$
Q 3 $|Q| = 3$

$n_2 = 1$ or 3 and $n_3 = 1$ or 4
if $n_3 = 4$ then $\exists$ 8 elements of order 3 ; hence $n_2 = 1$

$\Rightarrow$ P or Q is normal
$\Rightarrow$ $G \cong PQ$ is a non-trivial semidirect product
as non-abelian

$P \cong C_4$ or $C_2 \times C_2$; $Q \cong C_3$
$Aut(C_4) \cong C_2$, $Aut(C_2 \times C_2) \cong GL_2(\mathbb{Z}/2) \cong S_3$, $Aut(C_3) = C_2$

$P \trianglelefteq G$

$\phi : C_3 \to Aut(C_2)$ $\Rightarrow$ $\phi = id$ $\Rightarrow$ no non-abelian sd.
$\phi : C_3 \to Aut(C_2 \times C_2)$ $\Rightarrow$ $\phi \neq id$ is unique up to relabeling P

$\Rightarrow G \cong A_4 \cong C_3 \ltimes V_4$

$Q \trianglelefteq G$

$\phi : C_4 \to Aut(C_3)$ $\Rightarrow$ $\phi \neq id$ is unique

$\Rightarrow G \cong C_3 \rtimes C_4 = \langle x, y \mid x^3 = y^4 = e, yx = x^{-1}y \rangle$

$\phi : C_2 \times C_2 \to Aut(C_3)$ $\Rightarrow$ $\phi \neq id$ is unique upto relabeling P

$\Rightarrow G \cong D_{12} \cong C_6 \ltimes C_2 \cong (C_3 \times C_2) \ltimes C_2$
$\cong C_3 \ltimes (C_2 \times C_2)$

Note: The abelean groups of order 12 are: C_{12} and $C_6 \times C_2$

Problem: Determine all groups of order 99 up to isomorphism

Solution: $|G| = 99 = 3^2 \cdot 11$

$n_{11} \mid 9$, so $n_{11} = 1, 3, 9$

$n_{11} \equiv 1 \mod 11$, so $n_{11} = 1$

$n_3 \mid 11$, so $n_3 = 1, 11$

$n_3 \equiv 1 \mod 3$, so $n_3 = 1$

$\Longrightarrow$ both Sylow subgroups, P_{11}, P_3 are normal

$\Longrightarrow$ $P_3 P_{11} = G$ as $|P_3 P_{11}| = |P_3|\,|P_{11}| / |P_3 \cap P_{11}| = |G|$

$\Longrightarrow$ $G \simeq P_3 \times P_{11}$

$P_3 \simeq C_9$ or $P_3 \simeq C_3 \times C_3$; $P_{11} = C_{11}$

$\Longrightarrow$ G is $\simeq$

$C_9 \times C_{11}$ or $C_3 \times C_3 \times C_{11}$

Used: $G = N \rtimes_\varphi H$ and $H \lhd G$ $\Longleftrightarrow$ $G = N \times H$

<u>Problem</u>: Determine the group of order 66 up to isomorphism

<u>Solution</u>: $|G| = 66 = 2 \cdot 3 \cdot 11$;

let H Sylow 3-subgroup, Q Sylow 2-subgroup

K Sylow 11-subgroups ;

$n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 2 \cdot 3 \implies n_{11} = 1$;

hence K is normal ;

$\implies KH$ is a subgroup of G of order $3 \cdot 11$;

$3 \nmid 11-1 = 10 \implies KH \simeq C_3 \times C_{11} \simeq C_{33}$;

$|KH| = |G|/2 \implies KH$ is normal in G ;

$\implies (KH)Q = G$ is a semi direct product

determined by $\phi : C_2 \longrightarrow \mathrm{Aut}(C_3 \times C_{11})$

$\simeq C_3^\times \times C_{11}^\times$

i.e. an element of

order 2 in $C_3^\times \times C_{11}^\times \simeq C_2 \times C_{10}$

i.e. $(0,0)$; $(1,5)$; $(1,0)$; $(0,5)$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\implies G \simeq C_2 \times C_3 \times C_{11}$, D_{66} , $D_6 \times C_{11}$, $D_{22} \times C_3$

These are not isomorphic!

Problem: Assume $|G| = 132 = 11 \cdot 3 \cdot 2^2$.
Show that at least one of its Sylow subgroups is normal.

Solution: • $n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 3 \cdot 2^2 = 12$;

$\Rightarrow n_{11} = 1$ or $= 12$;

if $n_{11} = 12$ then there are $12 \cdot 10$ elements of order 11;

• $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 11 \cdot 2^2 = 44$;

$\Rightarrow n_3 = 1, 4, 44$

if $n_{11} = 12$ then $n_3 = 1$ or 4 as there are only 12 elements not of order 11

if $n_3 = 4$ then there are $2 \cdot 4$ elements of order 3

• $n_2 \equiv 1 \pmod 2$ and $n_2 \mid 11 \cdot 3$

$\Rightarrow n_2 = 1, 3, 11, 33$

if $n_{11} = 12$ and $n_3 = 4$ then there are only $132 - 120 - 8 = 4$ elements not of order 11 and 3

since the Sylow 2-subgroup must have order 4 these elements form the unique such group.