

As in lectures, (m, n) denotes the highest common factor of m and n .

1. Let a be a positive integer and suppose that in its decimal expansion it has 7 digits: $a = a_0 + 10a_1 + \cdots + 1000000a_6$. Show that a is divisible by 7 if and only if $a_0 + 3a_1 + 2a_2 - a_3 - 3a_4 - 2a_5 + a_6$ is divisible by 7.
2. Find a positive integer x such that $x \equiv 3 \pmod{4}$, $2x \equiv 5 \pmod{9}$ and $7x \equiv 1 \pmod{11}$.
3. Find the smallest positive integer x such that $x \equiv 11 \pmod{59}$ and $x \equiv 29 \pmod{71}$.
4. Show that $2^{340} \equiv 1 \pmod{341}$. Comment on this in connection with Fermat's Little Theorem.
5. Let $n = (6t+1)(12t+1)(18t+1)$ with $6t+1$, $12t+1$ and $18t+1$ all prime numbers. Prove that

$$a^{n-1} \equiv 1 \pmod{n}$$
 whenever $(a, n) = 1$. Comment on this in connection with Fermat's Little Theorem.
6. Show that if x is an integer then $x^{10} \in \{-1, 0, 1\} \pmod{25}$.
7. For which N is the following true: if you take an N digit number, reverse its digits and then add the result to the original number, you always get a multiple of 11?
8. Find all primes p for which the map $\phi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ defined by $\phi(x) = x^{13}$ is a group homomorphism.
9. Find all four-digit numbers N such that, when written in decimal, the last four digits of any power of N are the same as the digits of N .
10. For each of the following properties, show that there are infinitely many positive integers n which do *not* have that property.
 - (i) n is the sum of at most 3 squares;

- (ii) n is the sum of at most 8 sixth powers;
- (iii) n is the sum of at most 11 tenth powers;
- (iv) n is the sum of at most 15 fourth powers;
- (v) n is the sum of at most 7 (positive) seventh powers.

`ben.green@maths.ox.ac.uk`