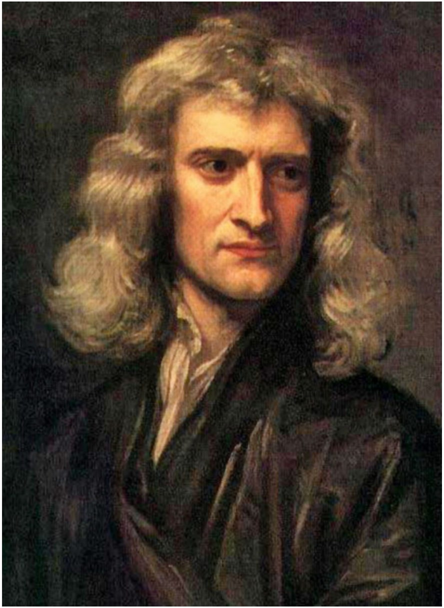
BO1 History of Mathematics
Lecture V
Newton's *Principia*

MT 2019 Week 3

Summary

- Isaac Newton (1642–1727)
- Kepler's laws, Descartes' theory, Hooke's conjecture
- The *Principia*
- Editions and influence of the *Principia*



Newton summarised

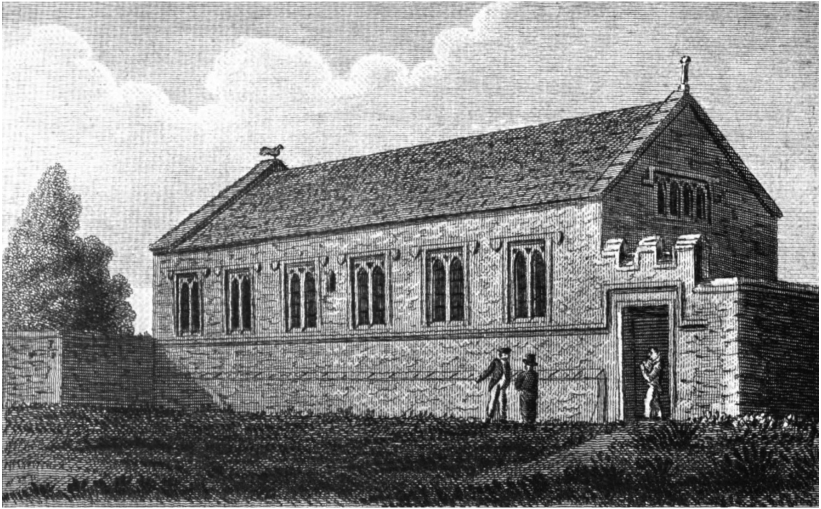
Alexander Pope, 1730:

Nature and Nature's Laws lay hid in Night.
God said, Let Newton be! and All was Light.

Woolsthorpe Manor

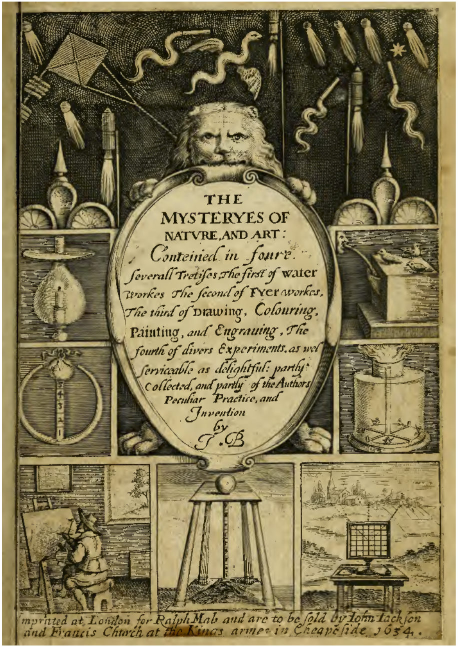
Newton born at Woolsthorpe Manor, 25th December 1642

Grantham Grammar School



Now The King's School, Grantham

John Bate: *The mysteries of nature and art* (1634)



John Bate: *The mysteries of nature and art* (1634)

little from it on the ſide Having thus prepared the barrell, fit a good thick board unto it, ſo that it may ſlip eaſily up and down from the top of the barrell unto the bottom, nayle a lether about the edges of it, and another upon the top of it: on the underſide of it let there be faſtned a good ſtiffe, but flexible ſpring of ſteele, which may thruſt the board from the bottom to the top of the barrell: let the foot of this ſpring reſt upon a barre faſtned acros the bottom of the barrell; let this board alſo have tied at the middle a little rope of length ſufficient. When you uſe it, bore a little hole in the table that you ſet it on, to put the rope thorow, and pull the rope down, which will contract the ſpring, and with it draw down the board: then poure in water at the baſin untill the veſſell be full: Note then, as you let ſlack the rope, the water will ſpirt out of the pipe, in the middle, and as you pull it ſtraight, the water will run into the veſſell againe. You may make birds, or divers images at the top of the pipe, out of which the water may break.

Another

Another manner of forcing water, whereby the water of any ſpring may be forced unto the top of a hill.

LEt there be two hollow poſts, with a ſuccur at the bottom of each, alſo a ſuccur nigh the top of each: let there be faſtned unto both theſe poſts a ſtrong peece of

timber, having, as it were, a beame or ſcale pinned in it, and having two handles, at each end one. In the tops of both

John Bate: *The mysteries of nature and art* (1634)

THE

SECOND BOOKE,

Teaching moſt plainly, and withall moſt exactly, the compoſing of all manner of Fire-works for Triumph and Recreation.

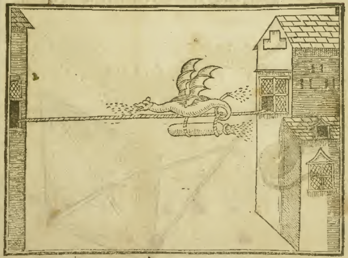
By *I. B.*

LONDON,

Printed by *Thomas Harper* for *Ralph Mab.* 1634.

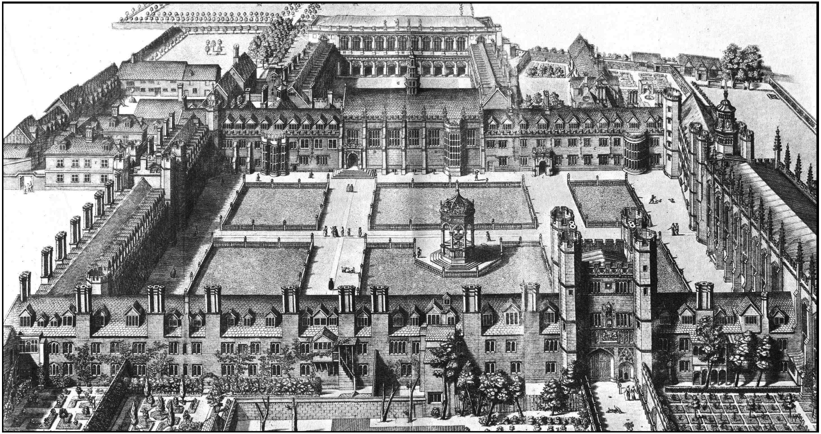
How to make flying Dragons.

THe flying Dragon is ſomewhat troubleſome to compoſe; it muſt be made eyther of dry and light wood, or crooked-lane plates, or of thin whalebones covered with Muſcovie glaſſe, and painted over. In the body thereof, there muſt bee a voyde cane to paſſe the rope through; unto the bottome of this cane muſt bee bound one or two large Rockets, according as the bigneſſe and



weight of the Dragon ſhall require; the body muſt bee filled with divers petrars, that may conſume it, and a ſparkling receipt muſt be ſo diſpoſed upon it, that being fired, it may burne both at the mouth and at the tayle thereof;

Trinity College, Cambridge



Isaac Newton (1642–1727)

Newton's major interests:

1650s:	model-building
1660s:	optics; (pure) mathematics
1670s:	alchemy, theology
1684+:	mathematics
1696–1727:	Warden of the Mint

Johannes Kepler (1571–1630)

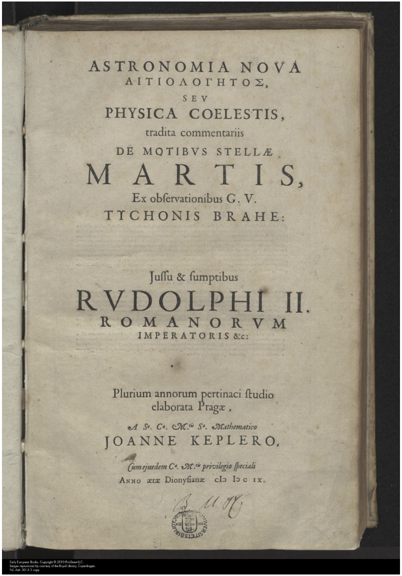
Engaged to sift through the astronomical data gathered by the Danish astronomer Tycho Brahe (1546–1601)

Major works:

Astronomia nova (1609)

Harmonices mundi (1619)

Kepler: *Astronomia nova* (1609)



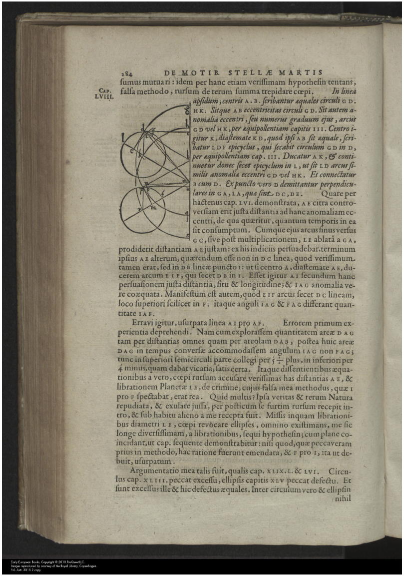
ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

Plurium annorum pertinaci ſtudio
elaborata Pragæ,

A S. C. M.tis S. Mathematico
JOANNE KEPLERO,

Cum ejusdem C. M.tis privilegio ſpeciali
ANNO æræ Dionyſianæ cIↃ IↃc IX.



ſumus mutuari: idem per hanc etiam veriſſimam hypotheſin tentans,
CAP. LVIII. falſa methodo, rurſum de rerum ſumma trepidare cœpi. *In linea apſidum, centris* A. B. *ſcribantur æquales circuli* GD. HK. *Sitque* AB *eccentricitas circuli* GD. *Sit autem anomalia eccentri, ſeu numerus graduum ejus, arcus* GD *vel* HK, *per æquipollentiam capitis* III. *Centro igitur* K, *diaſtemate* KD, *quod ipſi* AB *ſit æquale, ſcribatur* LDF *epicyclus, qui ſecabit circulum* GD *in* D, *per æquipollentiam cap.* III. *Ducatur* AK, *& continuetur donec ſecet epicyclum in* L, *ut ſit* LD *arcus ſimilis anomaliæ eccentri* GD *vel* HK. *Et connectatur* B *cum* D. *Ex puncto vero* D *demittantur perpendiculares in* GA, LA, *quæ ſint* DC, DE. Quare per hactenus cap. LVI. demonſtrata, AE citra controverſiam erit juſta diſtantia ad hanc anomaliam eccentri, de qua quæritur, quantum temporis in ea ſit conſumptum. Cumque ejus arcus ſinus verſus GC, ſive poſt multiplicationem, LE ablatâ a GA, prodiderit diſtantiam AE juſtam: ex his indiciis perſuadebar, terminum ipſius AE alterum, quærendum eſſe non in DC linea, quod veriſſimum tamen erat, ſed in DB lineæ puncto I: ut ſi centro A, diaſtemate AE, ducerem arcum EIF, qui ſecet DB in I. Eſſet igitur AI ſecundum hanc perſuaſionem juſta diſtantia, ſitu & longitudine; & IAG anomalia vere coæquata. Manifeſtum eſt autem, quod EIF arcus ſecet DC lineam, loco ſuperiori ſcilicet in F. itaque anguli IAG & FAG differant quantitate IAF.

Erravi igitur, uſurpata linea AI pro AF. Errorem primum experientia deprehendi. Nam cum exploraſſem quantitatem areæ DAG tam per diſtantias omnes quam per areolam DAB, poſtea huic areæ DAG in tempus converſæ accommodaſſem angulum IAG non FAG; tunc in ſuperiori ſemicirculi parte collegi per 5 $\frac{1}{2}$ plus, in inferiori per 4 minus, quam dabat vicaria, ſatis certa. Itaque diſſentientibus æquationibus a vero, cœpi rurſum accuſare veriſſimas has diſtantias AE, & librationem Planetæ LE, de crimine, cujus falſa mea methodus, quæ I pro F ſpectabat, erat rea. Quid multis? Ipſa veritas & rerum Natura repudiata, & exulare juſſa, per poſticum ſe furtim rurſum recepit intro, & ſub habitu alieno a me recepta fuit. Miſſis inquam librationibus diametri LE, cœpi revocare ellipſes, omnino exiſtimans, me ſic longe diverſiſſimam, a librationibus, ſequi hypotheſin; cum plane coincidant, ut cap. ſequente demonſtrabitur: niſi quod, quæ peccaveram prius in methodo, hac ratione fuerunt emendata, & F pro I, ita ut debuit, uſurpatum.

Argumentatio mea talis fuit, qualis cap. XLIX. L. & LVI. Circulus cap. XLIII. peccat exceſſu, ellipſis capitis XLV peccat defectu. Et ſunt exceſſus ille & hic defectus æquales. Inter circulum vero & ellipſin nihil

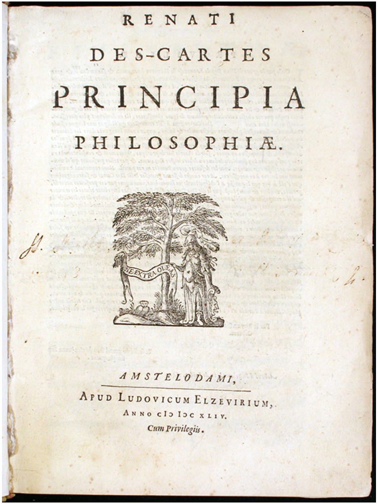
Kepler's laws

Kepler's laws of planetary motion (1609, 1619):

1. Planets move in elliptical orbits with the sun as focus
2. Planets sweep out equal areas in equal times
3. T^2 is proportional to R^3 (where T is time of one revolution, R is mean distance to sun)

All from empirical evidence

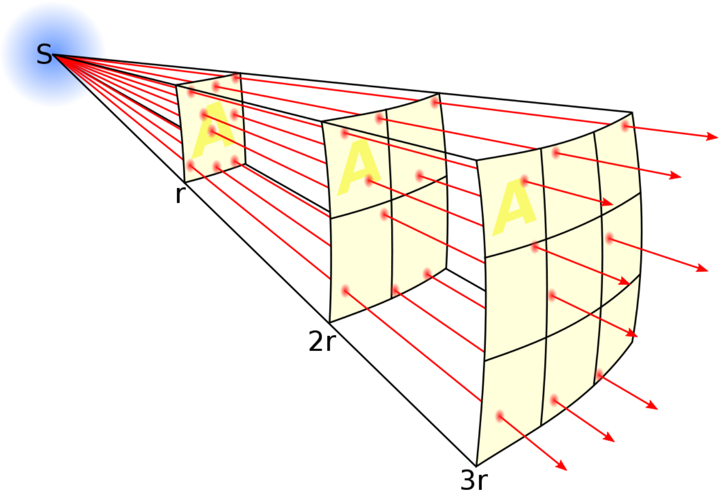
Descartes' theory



Descartes' views of planetary motion in *Principia philosophiae* (1644):

- the sun is one star among many
- asserted that planets are carried round their suns by vortices of the surrounding 'ether'
- claimed that theory could also explain magnetism and static electricity

An inverse square law?

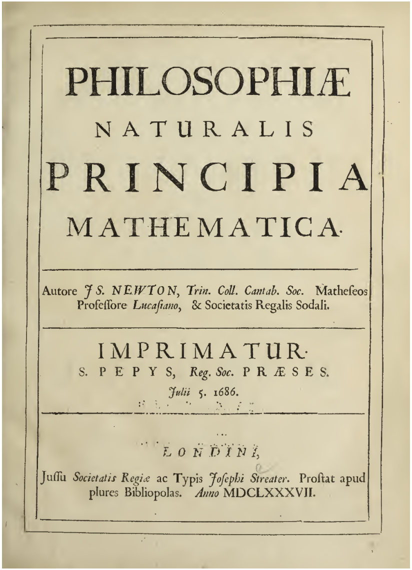


An inverse square law?

Speculations and calculations on an inverse square law of gravity:

1645	Ismaël Bullialdus refutes a claim of Kepler that 'gravity' drops off linearly with distance, instead suggesting an inverse square law
c. 1679	Hooke corresponds with Newton and suggests that an inverse square law might lead to elliptical orbits
1684	Halley visits Newton and asks whether this might be the case; Newton sends a short treatise on motion to Halley
1687	Publication of Newton's *Principia* at Halley's expense

Isaac Newton: *The mathematical principles of natural philosophy* (London, 1687)



PHILOSOPHIÆ

NATURALIS

PRINCIPIA

MATHEMATICA.

Autore *J S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.

S. PEPYS, *Reg. Soc.* PRÆSES.

Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Contents of the *Principia*

- Eight definitions — of matter, motion, innate force, impressed force, acceleration, ...
- Three axioms or Laws of Motion (as taught in school), together with six corollaries
- Book I: The motion of bodies
- Book II: The motion of bodies in resisting media
- Book III: The system of the world

The laws of motion

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perſeverare in ſtatu ſuo quieſcendi vel movendi uniformiter in directum, niſi quatenus a viribus impreſſis cogitur ſtatum illum mutare.

Projectilia perſeverant in motibus ſuis niſi quatenus a reſiſtentia aeris retardantur & vi gravitatis impelluntur deorſum. Trochus, cujus partes cohærendo perpetuo retrahunt ſeſe a motibus rectilineis, non ceſſat rotari niſi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus ſuos & progreſſivos & circulares in ſpatiis minus reſiſtentibus factos conſervant diutius.

Lex. II.

Mutationem motus proportionalem eſſe vi motrici impreſſæ, & fieri ſecundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, ſive ſimul & ſemel, ſive gradatim & ſucceſſive impreſſa fuerit. Et hic motus quoniam in eandem ſemper plagam cum vi generatrice determinatur, ſi corpus antea movebatur, motui ejus vel conſpiranti additur, vel contrario ſubducitur, vel oblique adjicitur, & cum eo ſecundum utriuſq; determinationem componitur.

Lex. III.

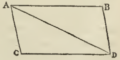
Lex. III.

Actioni contrariam ſemper & æqualem eſſe reactionem: ſive corporum duorum actiones in ſe mutuo ſemper eſſe æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; diſtentus eodem relaxandi ſe conatu urgebit Equum verſus lapidem, ac lapidem verſus equum, tantumq; impediet progreſſum unius quantum promovet progreſſum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi ſua quomodocunq; mutaverit, idem quoque viciſſim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem preſſionis mutuæ) ſubibit. His actionibus æquales fiunt mutationes non velocitatum ſed motuum, (ſcilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, ſunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore deſcribere, quo latera ſeparatis.



Si corpus dato tempore, vi ſola M, ferretur ab A ad B, & vi ſola N, ab A ad C, compleatur parallelogrammum $ABDC$, & vi utraq; feretur id eodem tempore ab A ad D. Nam quoniam vis N agit ſecundum lineam AC ipſi BD parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam BD a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam BD ſive vis N imprimatur, ſive non, atq; adeo in fine illius temporis reperietur alicubi in linea

illa

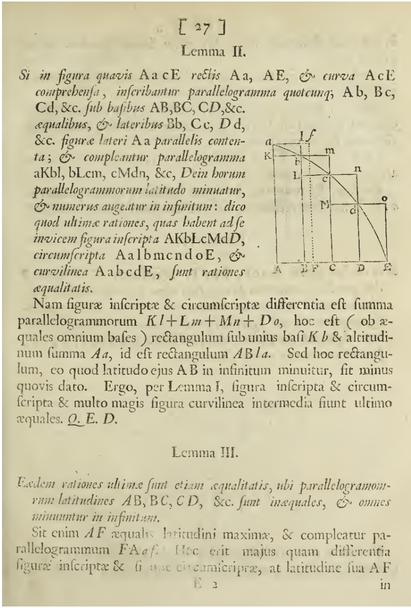
Book I: Motion of bodies

Book I, Section I: On the method of first and last ratios

Lemma I: Quantities, and ratios of quantities, which [...] approach nearer to each other than by any given difference, become ultimately equal.

For suppose they are ultimately unequal, and their ultimate difference is D. Then they cannot approach nearer to equality than by that difference.

Book I, Lemma II



Lemma II.

Si in figura quavis AacE *rectis* Aa, AE, *& curva* AcE *comprehensa, inscribantur parallelogramma quotcunq;* Ab, Bc, Cd, &c. *sub basibus* AB, BC, *CD*, &c. *æqualibus, & lateribus* Bb, Cc, *D*d, &c. *figuræ lateri* Aa *parallelis contenta; & compleantur parallelogramma* aKbl, bLcm, cMdn, &c, *Dein horum parallelogrammorum latitudo minuatur, & numerus augeatur in infinitum: dico quod ultimæ rationes, quas habent ad se invicem figura inscripta* AKbLcMd*D*, *circumscripta* AalbmcndoE, *& curvilinea* AabcdE, *sunt rationes æqualitatis.*

Nam figuræ inscriptæ & circumscriptæ differentia est summa parallelogrammorum $Kl + Lm + Mn + Do$, hoc est (ob æquales omnium bases) rectangulum sub unius basi Kb & altitudinum summa Aa, id est rectangulum $ABla$. Sed hoc rectangulum, eo quod latitudo ejus AB in infinitum minuitur, fit minus quovis dato. Ergo, per Lemma I, figura inscripta & circumscripta & multo magis figura curvilinea intermedia fiunt ultimo æquales. *Q. E. D.*

Lemma III.

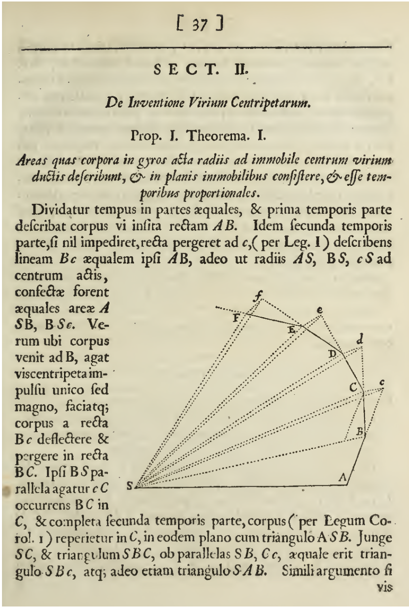
Eædem rationes ultimæ sunt etiam æqualitatis, ubi parallelogrammorum latitudines *A*B, B*C*, *C D*, &c. *sunt inæquales, & omnes minuuntur in infinitum.*

Sit enim *AF* æqualis latitudini maximæ, & compleatur parallelogrammum *F*A*af*. Hoc erit majus quam differentia figuræ inscriptæ & figuræ circumscriptæ, at latitudine sua AF

E 2 in

Lemma II: Ultimate equality of inscribed figure, circumscribed figure, and curved area

Motion under centripetal forces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SECT. II.

De Inventione Virium Centripetarum.

Prop. I. Theorema. I.

Areas quas corpora in gyros acta radiis ad immobile centrum virium ductis deſcribunt, & in planis immobilibus conſiſtere, & eſſe temporibus proportionales.

Dividatur tempus in partes æquales, & prima temporis parte deſcribat corpus vi inſita rectam *AB*. Idem ſecunda temporis parte, ſi nil impediret, recta pergeret ad *c*, (per Leg. I) deſcribens lineam *Bc* æqualem ipſi *AB*, adeo ut radiis *AS*, *BS*, *cS* ad centrum actis, confectæ forent æquales areæ *ASB*, *BSc*. Verum ubi corpus venit ad B, agat vis centripeta impulſu unico ſed magno, faciatq; corpus a recta *Bc* deflectere & pergere in recta *BC*. Ipſi *BS* parallela agatur *cC* occurrens *BC* in *C*, & completa ſecunda temporis parte, corpus (per Legum Corol. 1) reperietur in *C*, in eodem plano cum triangulo *ASB*. Junge *SC*, & triangulum *SBC*, ob parallelas *SB*, *Cc*, æquale erit triangulo *SBc*, atq; adeo etiam triangulo *SAB*. Simili argumento ſi vis

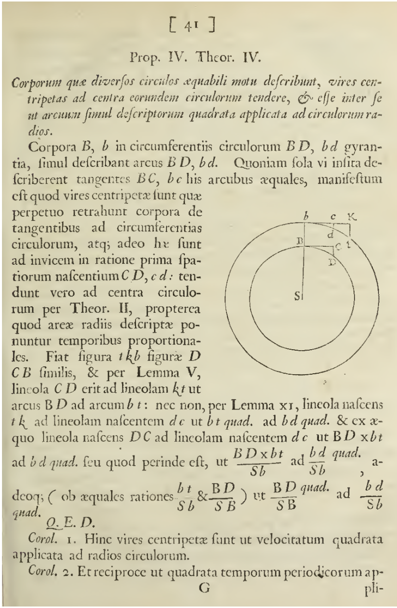
Book I, Section II: Motion under centripetal forces.

Proposition I: Bodies constrained by a central force to orbit a fixed point move in a plane and sweep out equal areas in equal times.

(Kepler's second law)

NB. independent of the 'law of force' involved.

Book I, Section II: Circular motion



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Prop. IV. Theor. IV.

Corporum quæ diverſos circulos æquabili motu deſcribunt, vires centripetas ad centra eorundem circulorum tendere, & eſſe inter ſe ut arcuum ſimul deſcriptorum quadrata applicata ad circulorum radios.

Corpora B, b in circumferentiis circulorum BD, bd gyrantia, ſimul deſcribant arcus BD, bd. Quoniam ſola vi inſita deſcriberent tangentes BC, bc his arcubus æquales, manifeſtum eſt quod vires centripetæ ſunt quæ perpetuo retrahunt corpora de tangentibus ad circumferentias circulorum, atq; adeo hæ ſunt ad invicem in ratione prima ſpatiorum naſcentium CD, cd: tendunt vero ad centra circulorum per Theor. II, propterea quod areæ radiis deſcriptæ ponuntur temporibus proportionales. Fiat figura tkb figuræ DCB ſimilis, & per Lemma V, lineola CD erit ad lineolam kt ut arcus BD ad arcum bt: nec non, per Lemma XI, lineola naſcens tk ad lineolam naſcentem dc ut bt *quad.* ad bd *quad.* & ex æquo lineola naſcens DC ad lineolam naſcentem dc ut $BD \times bt$ ad bd *quad.* ſeu quod perinde eſt, ut $\frac{BD \times bt}{Sb}$ ad $\frac{bd \; quad.}{Sb}$, adeoq; (ob æquales rationes $\frac{bt}{Sb}$ & $\frac{BD}{SB}$) ut $\frac{BD \; quad.}{SB}$ ad $\frac{bd}{Sb}$ *quad.*

Q. E. D.

Corol. 1. Hinc vires centripetæ ſunt ut velocitatum quadrata applicata ad radios circulorum.

Corol. 2. Et reciproce ut quadrata temporum periodicorum appli-

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Book I, Sect. II, Prop. IV: Motion under centripetal forces: motion in a circle.

Corollary 1: For motion in a circle centripetal force is proportional to $\dfrac{v^2}{r}$.

Corollary 6: For motion in a circle Kepler's third law implies an inverse square law of force.

Book I, Section III: orbits that are conic sections

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SECT. III.

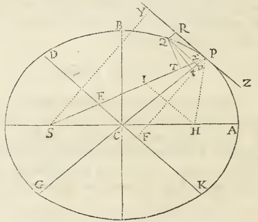
De motu Corporum in Conicis Sectionibus excentricis.

Prop. XI. Prob. VI.

Revolvatur corpus in Ellipsi: Requiritur lex vis centripetæ tendentis ad umbilicum Ellipseos.

Esto Ellipseos superioris umbilicus S. Agatur *SP* secans Ellipseos tum diametrum *DK* in *E*, tum ordinatim applicatam *Qv* in *x*, & compleatur parallelogrammum *Qx*P*R*. Patet *E*P æqualem esse semiaxi majori *AC*, eo quod acta ab altero Ellipseos umbilico H linea *HI* ipsi *EC* parallela, (ob æquales *CS*, *CH*) æquentur *ES*, *EI*, adeo ut *EP* semisumma sit ipsarum PS, P*I*, id est (ob parallelas *HI*, P*R* & angulos æquales *I*P*R*, *H*P*Z*) ipsorum PS, P*H*, quæ conjunctim axem totum 2*AC* adæquant. Ad SP demittatur perpendicularis *QT*, & Ellipseos latere recto principali (seu $\frac{2BC\ quad.}{AC}$) dicto *L*, erit $L \times QR$ ad $L \times Pv$ ut *QR* ad *Pv*; id est ut *PE* (seu *AC*) ad *PC*: & $L \times Pv$ ad *GvP* ut *L* ad *Gv*;

&

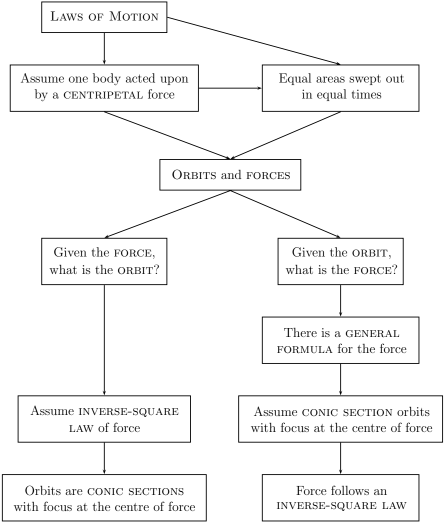


Proposition XI: Motion under centripetal forces: Kepler's First Law (orbit is an ellipse with sun at focus) implies an inverse square law of force.

Proposition XII: Motion under centripetal forces: hyperbolic orbit implies an inverse square law of force.

Proposition XIII: Motion under centripetal forces: parabolic orbit implies an inverse square law of force.

Book I, Sections II and III summarised



(Adapted from Colin Pask, *Magnificent Principia*, Prometheus Books, 2013, p. 218)

Book I, later sections

More mechanics of motion:

- converses: an inverse square law of force implies that orbits are conic sections;
- trajectories;
- much more besides.

All treated geometrically

Books II and III

Book II: Motion of bodies in resisting media:

Conclusion: "... it is manifest that the planets are not carried round in corporeal vortices ..." (Scholium to Proposition LIII)

Book III: The system of the world:

- Reconciliation of observation and theory
- Shape of the earth (correct?)
- Motion of the moon (wrong)
- Prediction of tides
- Comets

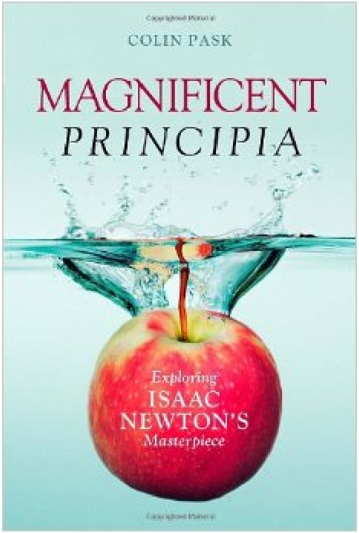
Influence of the *Principia*

Principia showed how mathematical methods could be used to study physical, especially but not exclusively, cosmological phenomena.

New ways of thinking: for example the method of ultimate ratios, though expressed geometrically, came close to a modern concept of limits.

Predictions could be verified by observation and experiment — verified (after some controversy) in the case of the shape of the earth, contradicted in the case of the motion of the moon.

For more on the *Principia*...



(Colin Pask, *Magnificent Principia*, Prometheus Books, 2013)

Three (very different) books among many...