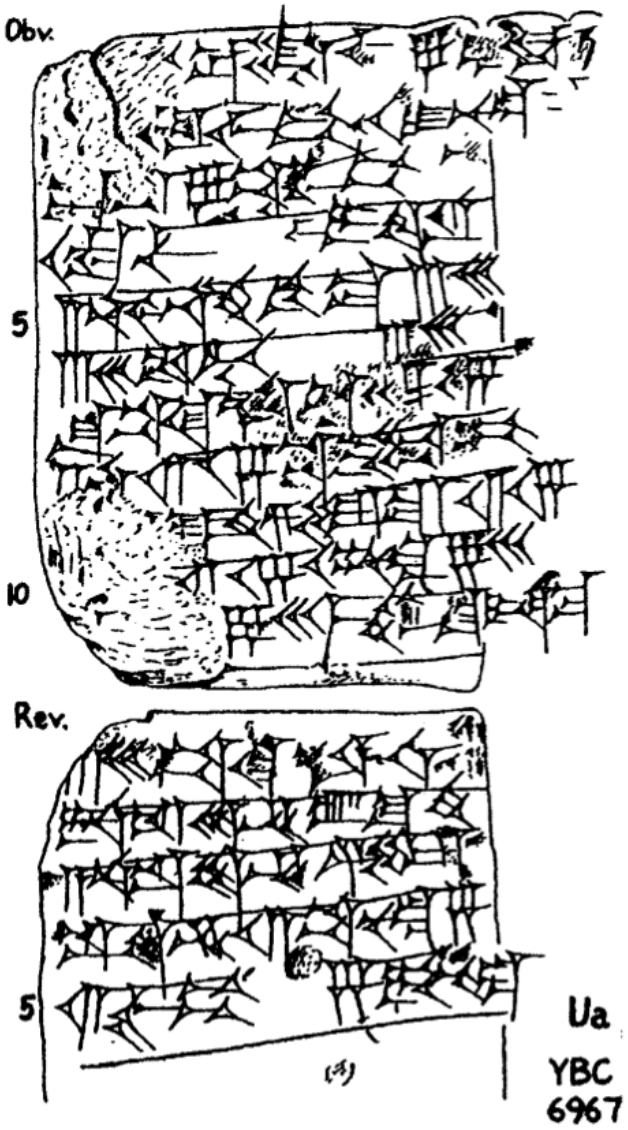
BO1 History of Mathematics
Lecture IX
Classical algebra: equation solving
1800BC – AD1800

MT 2019 Week 5

Summary

- Early quadratic equations
- Cubic and quartic equations
- Further 16th-century developments
- 17th century ideas
- 18th century ideas
- Looking back

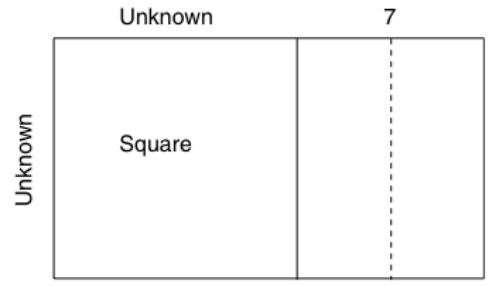
Completing the square, c. 1800 BC

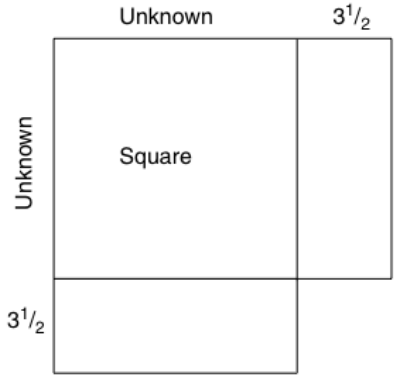


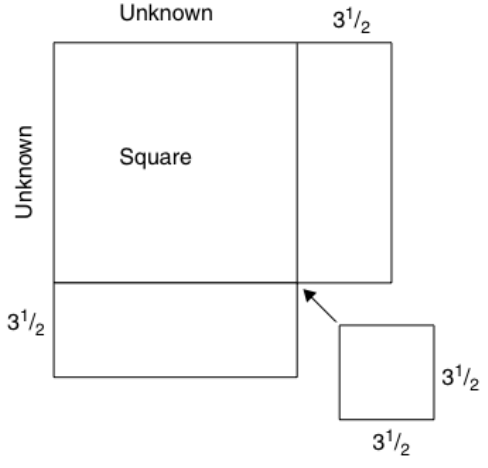
A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and $3\frac{1}{2}$ will come up. Multiply $3\frac{1}{2}$ by $3\frac{1}{2}$ and $12\frac{1}{4}$ will come up. Append 60, the area, to the $12\frac{1}{4}$ which came up for you and $72\frac{1}{4}$ will come up. What is the square-side of $72\frac{1}{4}$? $8\frac{1}{2}$. Put down $8\frac{1}{2}$ and $8\frac{1}{2}$ and subtract $3\frac{1}{2}$ from one of them; append $3\frac{1}{2}$ to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

Completing the square, c. 1800 BC







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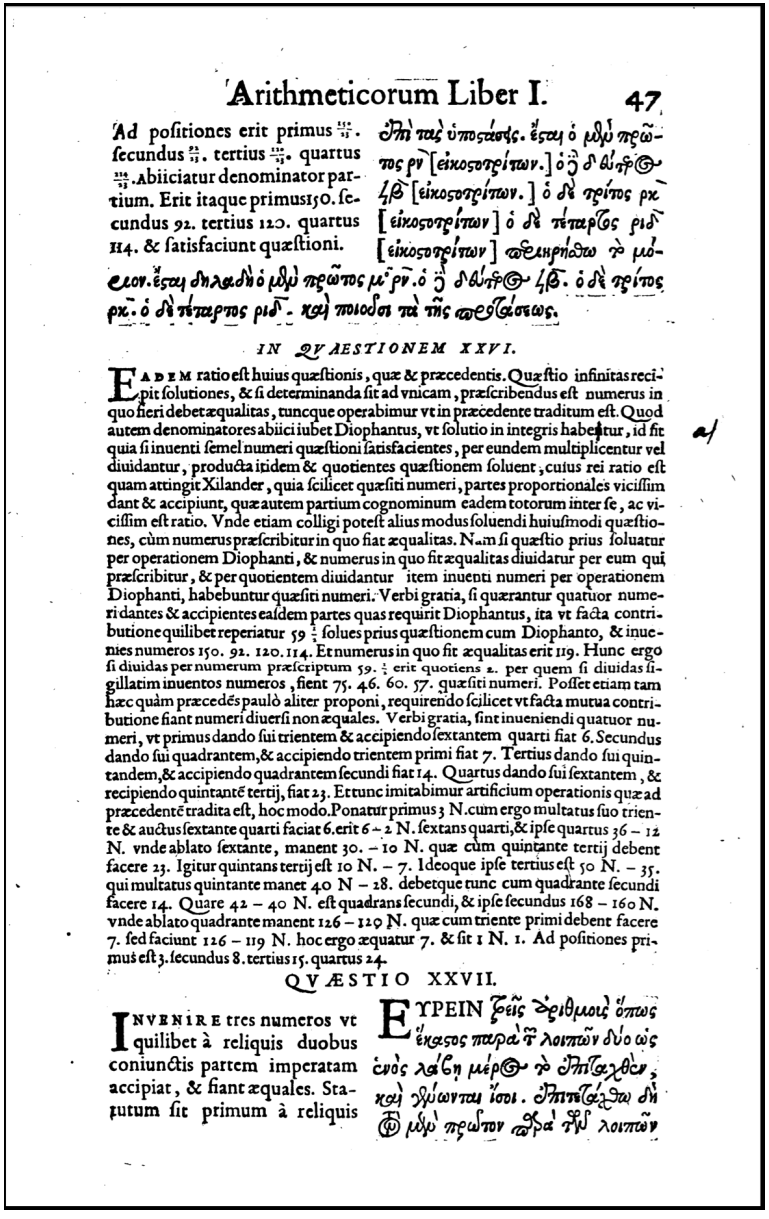
Points to note

- We have used the word ‘equation’ without writing down anything in symbols
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution — but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

Diophantus of Alexandria (3rd century AD)



Arithmeticorum Liber I. 47

Ad positiones erit primus $\frac{150}{23}$. secundus $\frac{92}{23}$. tertius $\frac{120}{23}$. quartus $\frac{114}{23}$. abiiciatur denominator partium. Erit itaque primus 150. secundus 92. tertius 120. quartus 114. & satisfaciunt quæstioni.

ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν πρῶτος ρν [εἰκοστοτρίτων.] ὁ δὲ δεύτερος ϛβ [εἰκοστοτρίτων.] ὁ δὲ τρίτος ρκ [εἰκοστοτρίτων] ὁ δὲ τέταρτος ριδ [εἰκοστοτρίτων] περιηρήσθω τὸ μόριον. ἔσται δηλαδὴ ὁ μὲν πρῶτος μ ρν. ὁ δὲ δεύτερος ϛβ. ὁ δὲ τρίτος ρκ. ὁ δὲ τέταρτος ριδ. καὶ ποιοῦσι τὰ τῆς προτάσεως.

IN QVAESTIONEM XXVI.

EADEM ratio est huius quæstionis, quæ & præcedentis. Quæstio infinitas recipit solutiones, & si determinanda sit ad vnicam, præscribendus est numerus in quo fieri debet æqualitas, tuncque operabimur vt in præcedente traditum est. Quod autem denominatores abiici iubet Diophantus, vt solutio in integris habeatur, id fit quia si inuenti semel numeri quæstioni satisfacientes, per eundem multiplicentur vel diuidantur, producta itidem & quotientes quæstionem soluent, cuius rei ratio est quam attingit Xilander, quia scilicet quæsiti numeri, partes proportionales vicissim dant & accipiunt, quæ autem partium cognominum eadem totorum inter se, ac vicissim est ratio. Vnde etiam colligi potest alius modus soluendi huiusmodi quæstiones, cùm numerus præscribitur in quo fiat æqualitas. Nam si quæstio prius soluatur per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui præscribitur, & per quotientem diuidantur item inuenti numeri per operationem Diophanti, habebuntur quæsiti numeri. Verbi gratia, si quærantur quatuor numeri dantes & accipientes easdem partes quas requirit Diophantus, ita vt facta contributione quilibet reperiatur 59 $\frac{1}{2}$ solues prius quæstionem cum Diophanto, & inuenies numeros 150. 92. 120. 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo si diuidas per numerum præscriptum 59. $\frac{1}{2}$ erit quotiens 2. per quem si diuidas sigillatim inuentos numeros, fient 75. 46. 60. 57. quæsiti numeri. Posset etiam tam hæc quàm præcedēs paulò aliter proponi, requirendo scilicet vt facta mutua contributione fiant numeri diuersi non æquales. Verbi gratia, sint inueniendi quatuor numeri, vt primus dando sui trientem & accipiendo sextantem quarti fiat 6. Secundus dando sui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando sui quintandem, & accipiendo quadrantem secundi fiat 14. Quartus dando sui sextantem, & recipiendo quintantē tertij, fiat 23. Et tunc imitabimur artificium operationis quæ ad præcedentē tradita est, hoc modo. Ponatur primus 3 N. cum ergo multatus suo triente & auctus sextante quarti faciat 6. erit 6 – 2 N. sextans quarti, & ipse quartus 36 – 12 N. vnde ablato sextante, manent 30. – 10 N. quæ cùm quintante tertij debent facere 23. Igitur quintans tertij est 10 N. – 7. Ideoque ipse tertius est 50 N. – 35. qui multatus quintante manet 40 N – 28. debetque tunc cum quadrante secundi facere 14. Quare 42 – 40 N. est quadrans secundi, & ipse secundus 168 – 160 N. vnde ablato quadrante manent 126 – 120 N. quæ cum triente primi debent facere 7. sed faciunt 126 – 119 N. hoc ergo æquatur 7. & sit 1 N. 1. Ad positiones primus est 3. secundus 8. tertius 15. quartus 24.

QVÆSTIO XXVII.

INVENIRE tres numeros vt quilibet à reliquis duobus coniunctis partem imperatam accipiat, & fiant æquales. Statutum sit primum à reliquis

ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως ἕκαστος παρὰ τῶν λοιπῶν δύο ὡς ἑνὸς λάβῃ μέρος τὸ ἐπιταχθὲν, καὶ γένωνται ἴσοι. ἐπιτετάχθω δὴ τῷ μὲν πρώτου παρὰ τῶν λοιπῶν

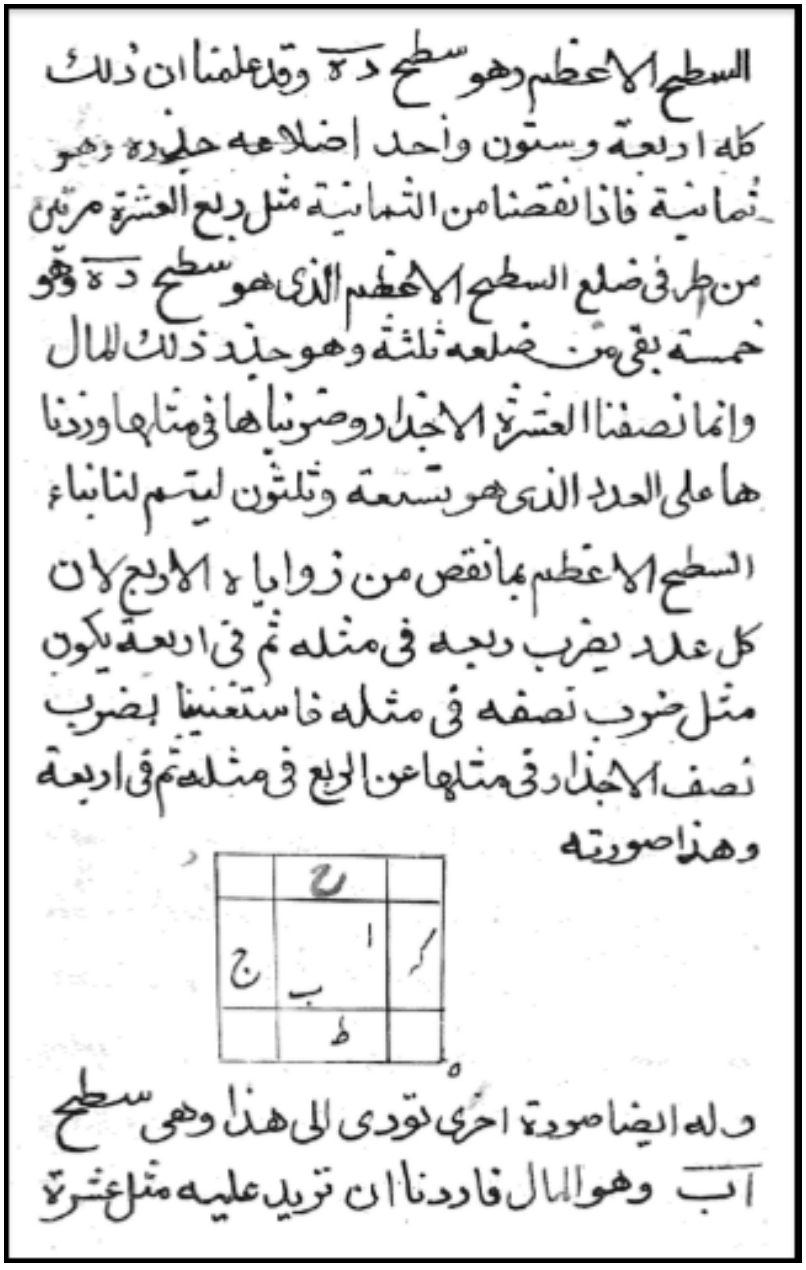
Problem I.27: Find two numbers such that their sum and product are given numbers

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:

1. Squares are equal to roots ($ax^2 = bx$)
2. Squares are equal to numbers ($ax^2 = c$)
3. Roots are equal to numbers ($bx = c$)
4. Squares and roots are equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers are equal to roots ($ax^2 + c = bx$)
6. Roots and numbers are equal to squares ($bx + c = ax^2$)

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

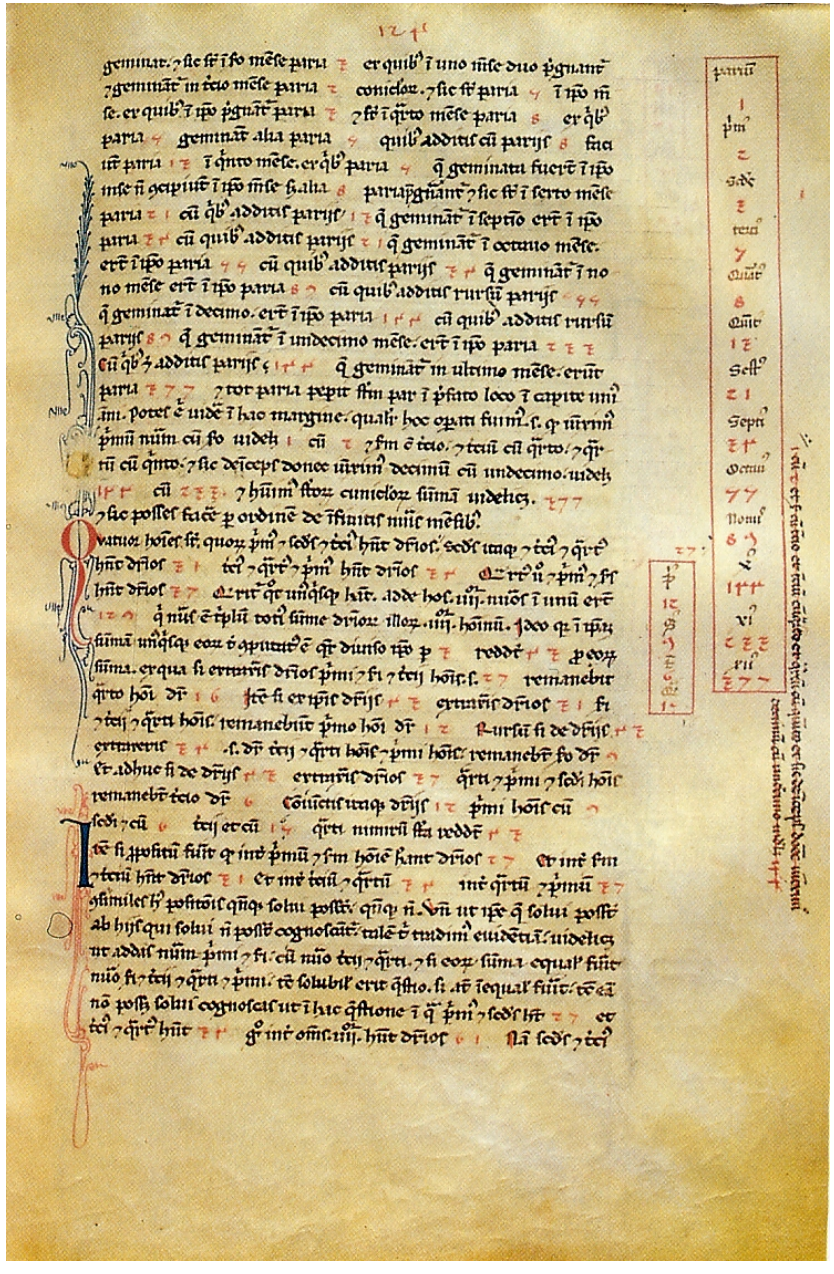

السطح الاعظم وهو سطح د ه وقد علمنا ان ذلك
كله اربعة وستون واحد اضلاعه جذره وهو
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين
من طرفي ضلع السطح الاعظم الذي هو سطح د ه وهو
خمسة بقي من ضلعه ثلثة وهو حينئذ ذلك المال
وانما نصفنا العشرة الاجذار وضربناها في مثلها وردنا
ها على العدد الذي هو تسعة وثلثون ليتم لنا بناء
السطح الاعظم بانقص من زوايا ه الاربع لان
كل عدد يضرب ربعه في مثله ثم في اربعة يكون
مثل ضرب نصفه في مثله فاستغنينا بضرب
نصف الاجذار في مثلها عن الربع في مثله ثم في اربعة
وهذا صورته
ح
ا
ك
ج
ب
ط
وله ايضا صورة اخرى تؤدي الى هذا وهي سطح
آب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on the previous slide

Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

Liber abaci (or *Liber abbaci*), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



Cubic equations (1)

Italy, early 16th century:

solutions to cubics of the form $x^3 + px = q$

- found by Scipione del Ferreo (or Ferro) (c. 1520)
- taught to Antonio Maria Fiore (pupil)
- and Annibale della Nave (son-in-law)
- rediscovered by Niccolò Tartaglia (1535)
- passed in rhyme to Girolamo Cardano (1539)

Cubic equations (2)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

Cubic equations (3)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

Interpretation of Tartaglia's rhyme:

Find u, v such that

$$u - v = q, \quad uv = \left(\frac{p}{3}\right)^3.$$

Then

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

NB: In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

with similar expressions (in radicals) for the remaining two roots

Cardano's *Ars magna, sive de regulis algebraicis* (1545)

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

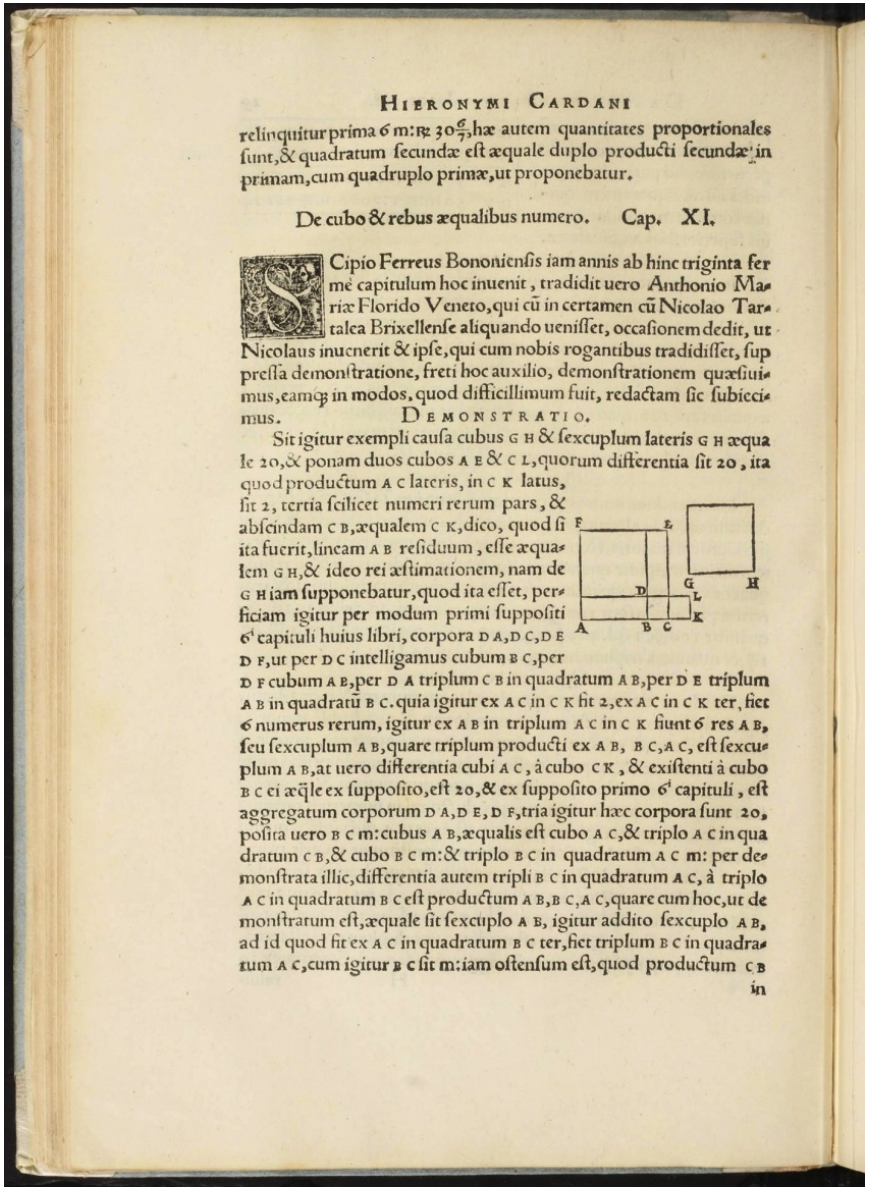
HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

3 Numerus & positio, & ignota quantitas.
4 Numerus & q̃dratũ positionis, & ignota quantitas, seu numerus & q̃d^tum quantitatis ignotæ & positio, seu numerus cũ q̃d^to positionis quantitatis ignotæ, seu numerus & productum ex positione in quantitatem ignotam, cum altera earum, uel cum q̃drato unius earum.

Capitula primitiua.	Capitula deriuatiua.
1 Numerus æqualis q̃d^to & rebus.	1 Numerus æq̃lis q̃d' q̃d' & q̃d'.
	2 Numerus æqualis cu' q̃d' & cub'.
2 Numerus & res æqualia quadrato.	3 Numerus & q̃d' æquales q̃d' q̃d'.
	4 Numerus & cub' æq̃les cub' q̃d'.
3 Numerus & q̃d' æqualia rebus.	5 Numerus & q̃d' q̃d' æqualia q̃d'.
	6 Numerus & cub' q̃d' æq̃lia cub'.
4 Numerus æqualis cubo & rebus.	7 Numerus æq̃lis q̃d' & cub' q̃d'.
	8 Numerus æq̃lis cub' & cubo cubi.
5 Numerus & res æqualia cubis.	9 Numerus & q̃d' æq̃lia cub' q̃d'.
	10 Numerus & cub' æq̃lia cub' cub'.
6 Numerus & cubus æq̃l' rebus æq̃tio prima.	11 Numerus & cub' q̃d' æq̃l' q̃d' æq̃° pri^a.
	12 Nu^r' & cub' cub' æq̃lia cub' æq̃° pri^a.
7 Numerus & cub' æq̃lia rebus æq̃° secunda.	13 Nu^r' & cub' q̃d' æq̃lia q̃d' æq̃° secunda.
	14 Nu^r' & cub' cub' æq̃lia cub' æq̃° secũd^a.
8 Numerus æq̃lis q̃drato & cubo.	15 Numerus æq̃lis q̃d' q̃d' & cub' q̃drati.
	16 Numerus æq̃lis cub' q̃d' & cub' cubi.
9 Numerus & q̃dratum æqualia cubo.	17 Numerus & q̃d' q̃d' æq̃lia cub' q̃drati.
	18 Numerus & cub' q̃d' æq̃lia cub' cubi.
10 Numerus & cub' æq̃lia q̃drato æq̃° prima.	19 Nu^r' & cub' q̃d' æq̃lia q̃d' q̃d' æq̃° pri^a.
	20 Nu^r' & cub' cub' æq̃lia cub' q̃d' æq̃° pri^a.
11 Numerus & cub' æq̃lia q̃drato æq̃°secunda.	21 Nu^r' & cub' q̃d' æq̃l' q̃d' q̃d' æq̃° secũd^a.
	22 Nu^r' & cub' cub' æq̃l' cu' q̃d' æq̃° secũd.
12 Numerus æq̃lis rebus q̃drato & cubo.	23 Nu^r' æq̃lis q̃d' & q̃d' q̃d' & cub' q̃d'.
	24 Nu^r' æq̃lis cub' & cub' q̃d' & cub' cu'.
13 Numerus & res æq̃lia quadrato & cubo.	25 Nu^r' & q̃d' æq̃lia q̃d' q̃d' & cub' q̃d'.
	26 Nu^r' & cub' æq̃lia cub' q̃d & cub' cu'.
14 Numerus & res & q̃d' æqualia cubo.	27 Nu^r' & q̃d' & q̃d' q̃d' æq̃lia cub' q̃d'.
	28 Nu^r' & cub' & cub q̃d æq̃lia cu cub'.
15 Numerus & q̃d' æq̃lia rebus & cub' æq̃° prima.	29 Nu^r' & q̃d' q̃d' æq̃l' q̃d' & cu q̃d æq̃° p^a
	30 Nu^r' & cu' q̃d' æq̃l' cu & cu' cu æq̃° pri^a.
16 Numerus & q̃d' æq̃lia rebus & cubo æq̃° secũda.	31 Nu^r' & q̃d' q̃d' æq̃l' q̃d' & cu' q̃d æq̃° sec.
	32 Nu^r' & cu' q̃d' æq̃l' cu' & cu' cu' æq̃° sec.

17 Nume-

Cardano on the cubic



HIERONYMI CARDANI

relinquitur prima 6 m: R: 30 6/7, hæ autem quantitates proportionales sunt, & quadratum secundæ est æquale duplo producti secundæ in primam, cum quadruplo primæ, ut proponebatur.

De cubo & rebus æqualibus numero. Cap. XI.

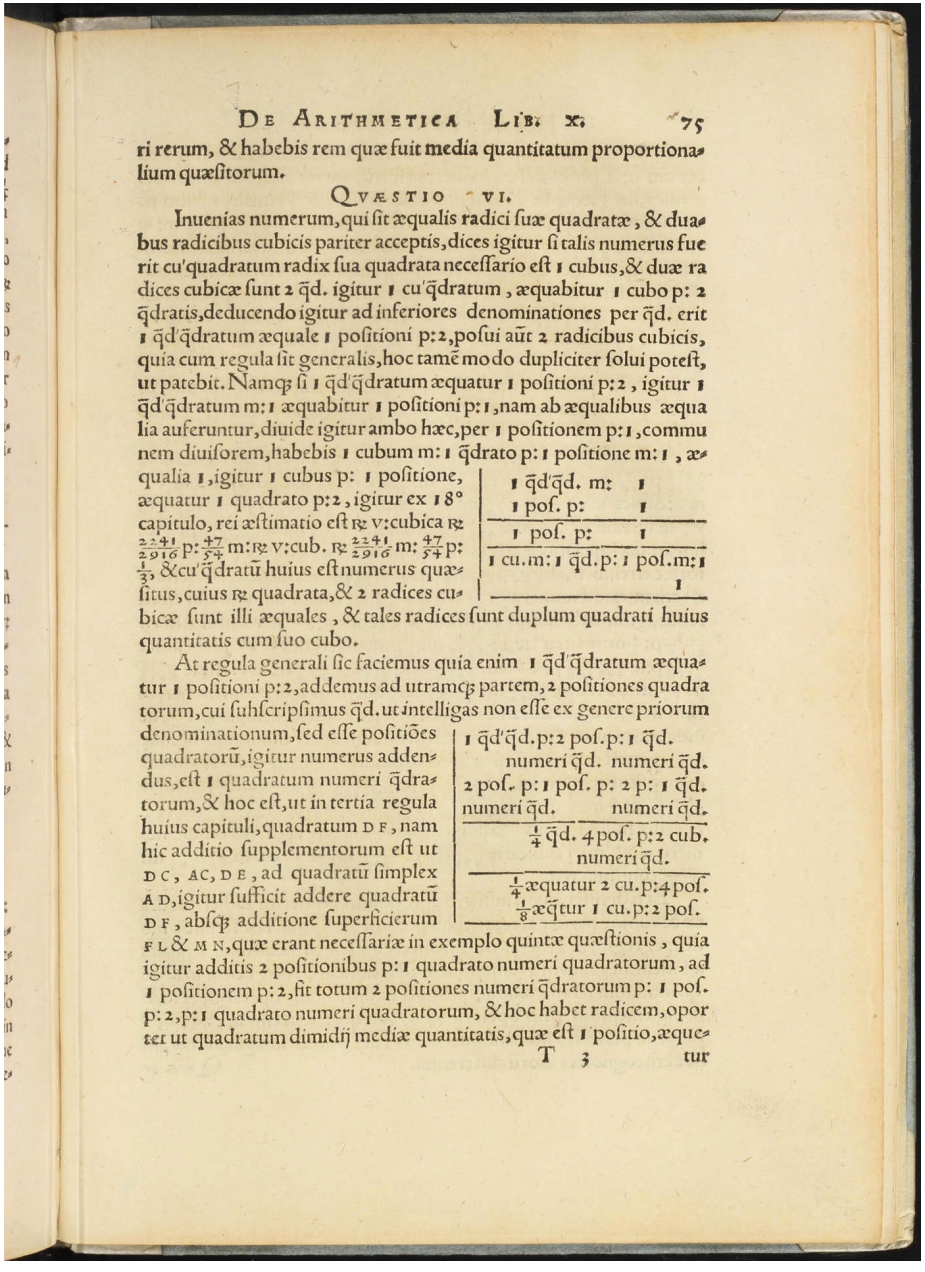
SCipio Ferreus Bononiensis iam annis ab hinc triginta fer mè capitulum hoc inuenit, tradidit uero Anthonio Ma riæ Florido Veneto, qui cũ in certamen cũ Nicolao Tar talea Brixellense aliquando uenisset, occasionem dedit, ut Nicolaus inuenerit & ipse, qui cum nobis rogantibus tradidisset, sup pressa demonstratione, freti hoc auxilio, demonstrationem quæsiui mus, eamq; in modos, quod difficillimum fuit, redactam sic subicci mus. DEMONSTRATIO.

Sit igitur exempli causa cubus G H & sexcuplum lateris G H æqua le 20, & ponam duos cubos A E & C L, quorum differentia sit 20, ita quod productum A C lateris, in C K latus, sit 2, tertia scilicet numeri rerum pars, & abscindam C B, æqualem C K, dico, quod si ita fuerit, lineam A B residuum, esse æqua lem G H, & ideo rei æstimationem, nam de G H iam supponebatur, quod ita esset, per ficiam igitur per modum primi suppositi 6^i capituli huius libri, corpora D A, D C, D E D F, ut per D C intelligamus cubum B C, per D F cubum A B, per D A triplum C B in quadratum A B, per D E triplum A B in quadratũ B C. quia igitur ex A C in C K fit 2, ex A C in C K ter, fiet 6 numerus rerum, igitur ex A B in triplum A C in C K fiunt 6 res A B, seu sexcuplum A B, quare triplum producti ex A B, B C, A C, est sexcuplum A B, at uero differentia cubi A C, à cubo C K, & existenti à cubo B C ei æq̃le ex supposito, est 20, & ex supposito primo 6^i capituli, est aggregatum corporum D A, D E, D F, tria igitur hæc corpora sunt 20, posita uero B C m: cubus A B, æqualis est cubo A C, & triplo A C in qua dratum C B, & cubo B C m: & triplo B C in quadratum A C m: per de monstrata illic, differentia autem tripli B C in quadratum A C, à triplo A C in quadratum B C est productum A B, B C, A C, quare cum hoc, ut de monstratum est, æquale sit sexcuplo A B, igitur addito sexcuplo A B, ad id quod fit ex A C in quadratum B C ter, fiet triplum B C in quadra tum A C, cum igitur B C sit m: iam ostensum est, quod productum C B in

- Geometrical justification remains
- General solution (to particular case), rather than example to be followed
- Make substitution $x = y - \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic



DE ARITHMETICA LIB. X. 75

ri rerum, & habebis rem quæ fuit media quantitatum proportionalium quæsitorum.

QVÆSTIO VI.

Inuenias numerum, qui sit æqualis radici suæ quadratæ, & duabus radicibus cubicis pariter acceptis, dices igitur si talis numerus fuerit cu'quadratum radix sua quadrata necessario est 1 cubus, & duæ radices cubicæ sunt 2 q̃d. igitur 1 cu'q̃dratum, æquabitur 1 cubo p: 2 q̃dratis, deducendo igitur ad inferiores denominationes per q̃d. erit 1 q̃d'q̃dratum æquale 1 positioni p: 2, posui aũt 2 radicibus cubicis, quia cum regula sit generalis, hoc tamẽ modo dupliciter solui potest, ut patebit. Namq; si 1 q̃d'q̃dratum æquatur 1 positioni p: 2, igitur 1 q̃d'q̃dratum m: 1 æquabitur 1 positioni p: 1, nam ab æqualibus æqualia auferuntur, diuide igitur ambo hæc, per 1 positionem p: 1, communem diuisorem, habebis 1 cubum m: 1 q̃drato p: 1 positione m: 1, æqualia 1, igitur 1 cubus p: 1 positione, æquatur 1 quadrato p: 2, igitur ex 18° capitulo, rei æstimatio est ℞ v: cubica ℞ $\frac{2241}{2916}$ p: $\frac{47}{54}$ m: ℞ v: cub. ℞ $\frac{2241}{2916}$ m: $\frac{47}{54}$ p: $\frac{1}{3}$, & cu'q̃dratũ huius est numerus quæsitus, cuius ℞ quadrata, & 2 radices cubicæ sunt illi æquales, & tales radices sunt duplum quadrati huius quantitatis cum suo cubo.

1 q̃d'q̃d. m:	1
1 pos. p:	1
1 pos. p:	1
1 cu. m: 1 q̃d. p: 1 pos. m:	1

At regula generali sic faciemus quia enim 1 q̃d'q̃dratum æquatur 1 positioni p: 2, addemus ad utramq; partem, 2 positiones quadratorum, cui subscripsimus q̃d. ut intelligas non esse ex genere priorum denominationum, sed esse positiões quadratorũ, igitur numerus addendus, est 1 quadratum numeri q̃dratorum, & hoc est, ut in tertia regula huius capituli, quadratum D F, nam hic additio supplementorum est ut D C, AC, D E, ad quadratũ simplex A D, igitur sufficit addere quadratũ D F, absq; additione superficierum F L & M N, quæ erant necessariæ in exemplo quintæ quæstionis, quia igitur additis 2 positionibus p: 1 quadrato numeri quadratorum, ad 1 positionem p: 2, fit totum 2 positiones numeri q̃dratorum p: 1 pos. p: 2, p: 1 quadrato numeri quadratorum, & hoc habet radicem, oportet ut quadratum dimidij mediæ quantitatis, quæ est 1 positio, æque-

1 q̃d'q̃d. p: 2 pos. p: 1 q̃d. numeri q̃d.	numeri q̃d.
2 pos. p: 1 pos. p: 2 p: 1 q̃d. numeri q̃d.	numeri q̃d.
$\frac{1}{4}$ q̃d. 4 pos. p: 2 cub. numeri q̃d.	
$\frac{1}{4}$ æquatur 2 cu. p: 4 pos.	
$\frac{1}{8}$ æq̃tur 1 cu. p: 2 pos.	

Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

Add $2yx^2 + y^2$ to each side to give

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2).$$

Now we seek y such that the right hand side is a perfect square:

$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

NB: In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's *Ars Magna* may also be found online here

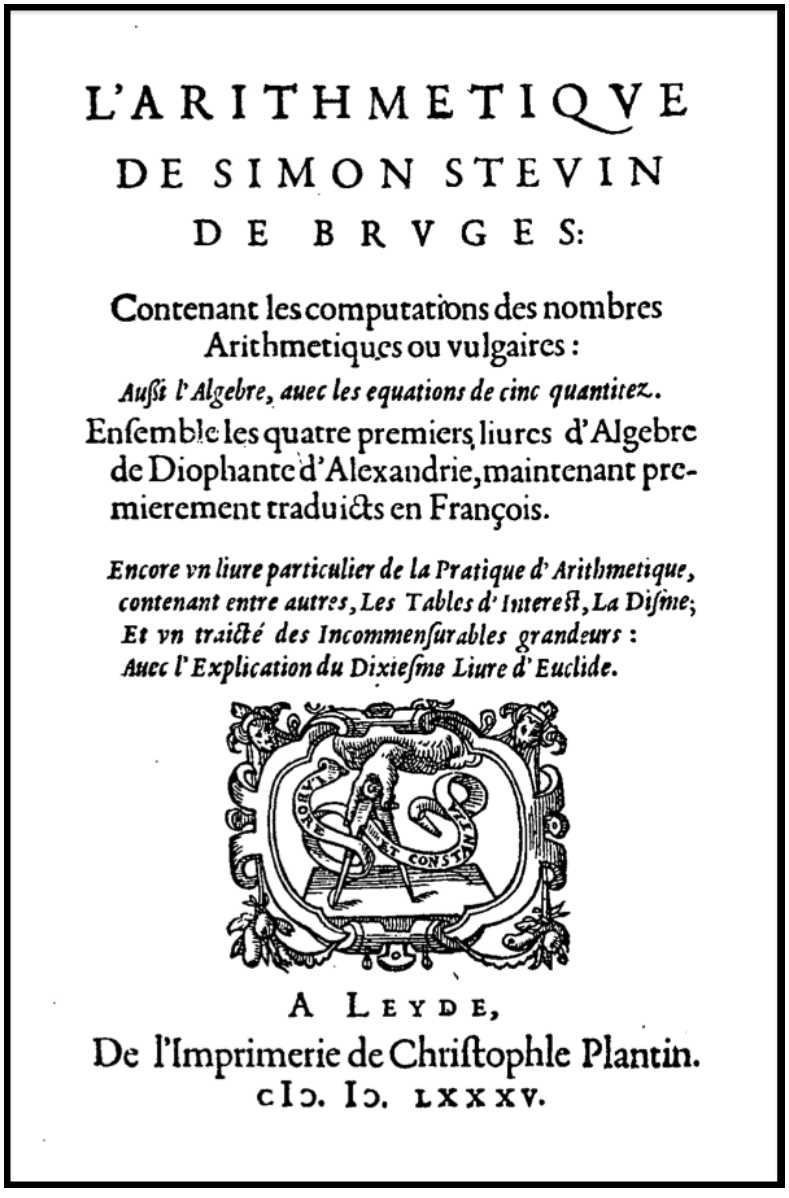
Further 16th-century developments

Rafael Bombelli, *L'algebra* (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers

  [to be dealt with in a later lecture]

Further 16th-century developments



L'ARITHMETIQVE
DE SIMON STEVIN
DE BRVGES:

Contenant les computations des nombres Arithmetiques ou vulgaires :
Auſſi l'Algebre, auec les equations de cinc quantitez.
Enſemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.

Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Intereſt, La Diſme; Et vn traicté des Incommenſurables grandeurs : Auec l'Explication du Dixieſme Liure d'Euclide.

A LEYDE,
De l'Imprimerie de Chriſtophle Plantin.
cIↃ. IↃ. LXXXV.

Simon Stevin, *L'arithmetique ... aussi l'algebre* (1585):

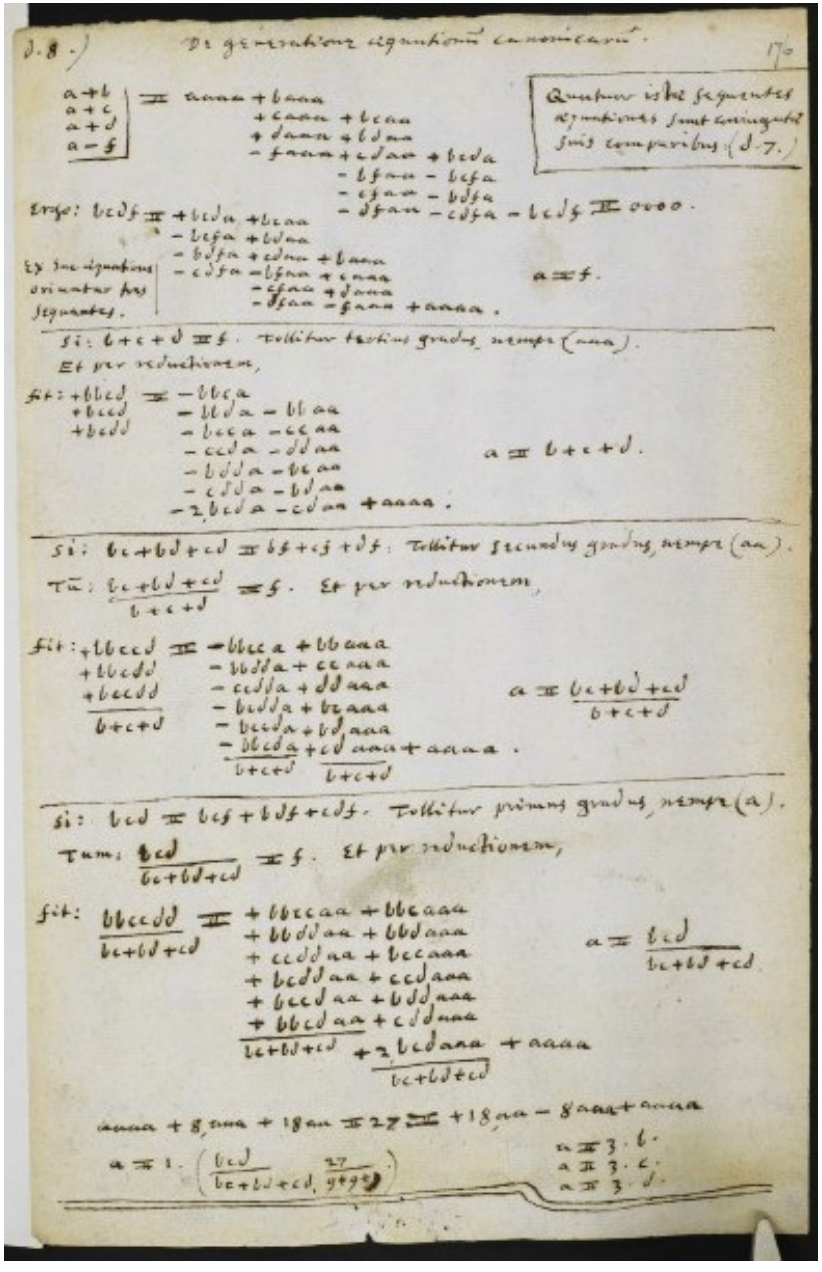
- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations

Thomas Harriot (c. 1600)



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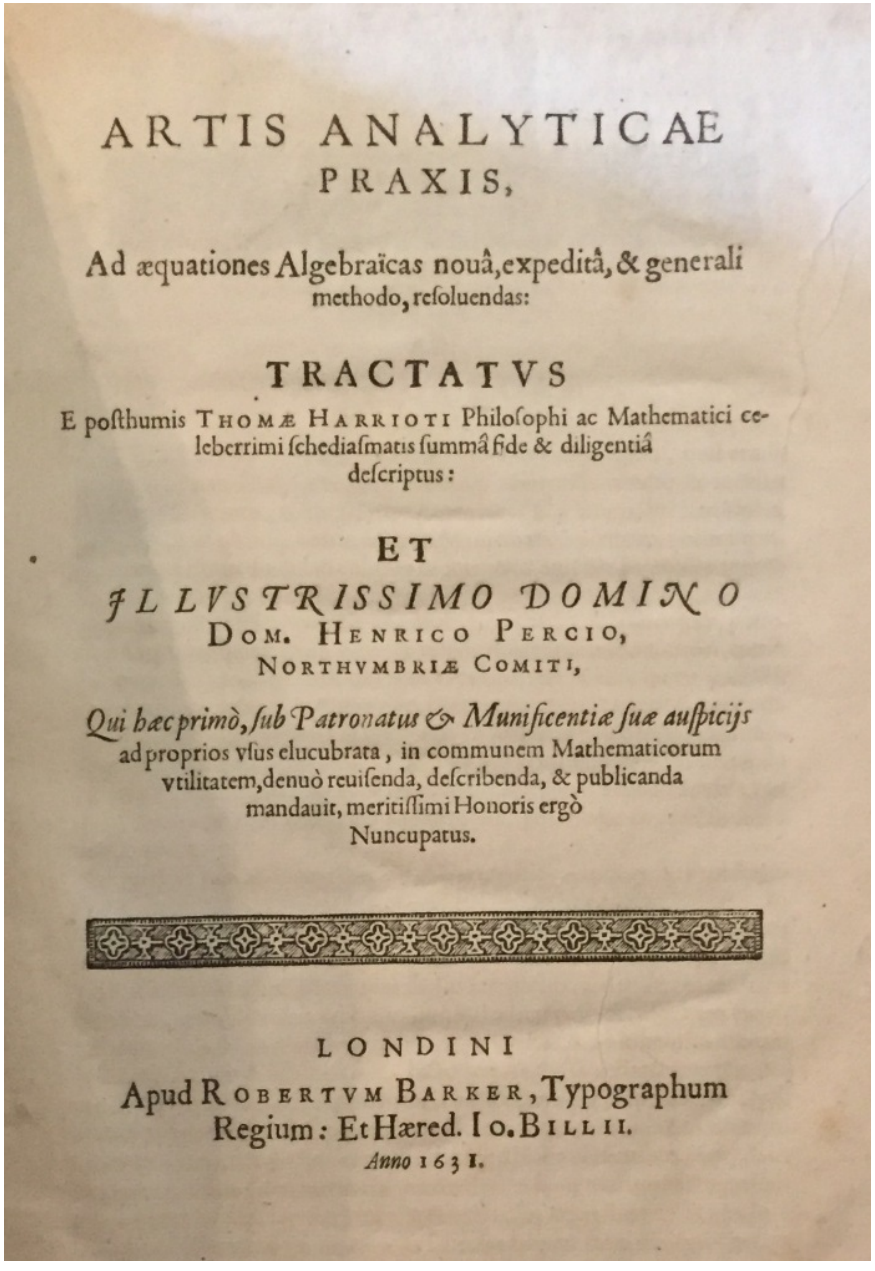
Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, reſoluendas:

TRACTATVS

E poſthumis THOMÆ HARRIOTI Philoſophi ac Mathematici celeberrimi ſchediaſmatis ſummâ fide & diligentiâ deſcriptus:

ET

ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

Qui hæc primò, ſub Patronatus & Munificentiæ ſuæ auſpicijs ad proprios vſus elucubrata, in communem Mathematicorum vtilitatem, denuò reuiſenda, deſcribenda, & publicanda mandauit, meritiſſimi Honoris ergò Nuncupatus.

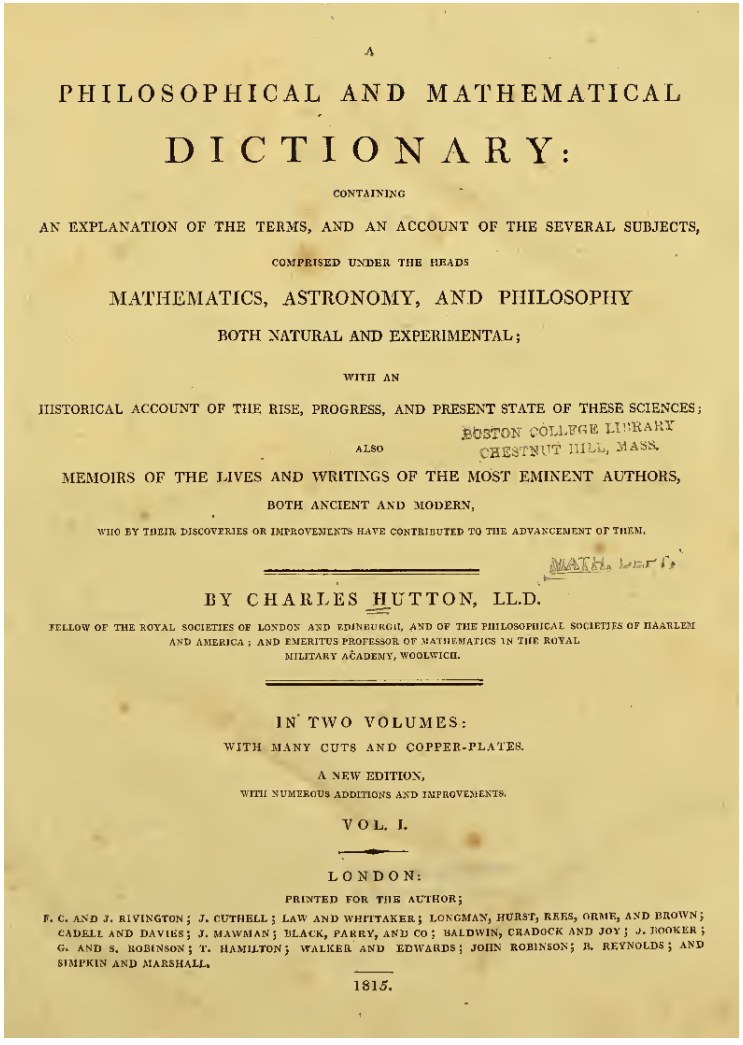
LONDINI
Apud ROBERTVM BARKER, Typographum Regium: Et Hæred. Io. BILLII.
Anno 1631.

Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

Commentary on Harriot



A

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1815.

Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; . . .

Algebra in the 17th century

From 1600 onwards, 'algebra' as a set of recipes and techniques began to diverge in two (linked) directions:

- 'algebra' as a tool or a language (a.k.a. 'analysis' or the 'analytic art')
- 'algebra' as an object of study in its own right (the 'theory of equations')

Descartes on algebra

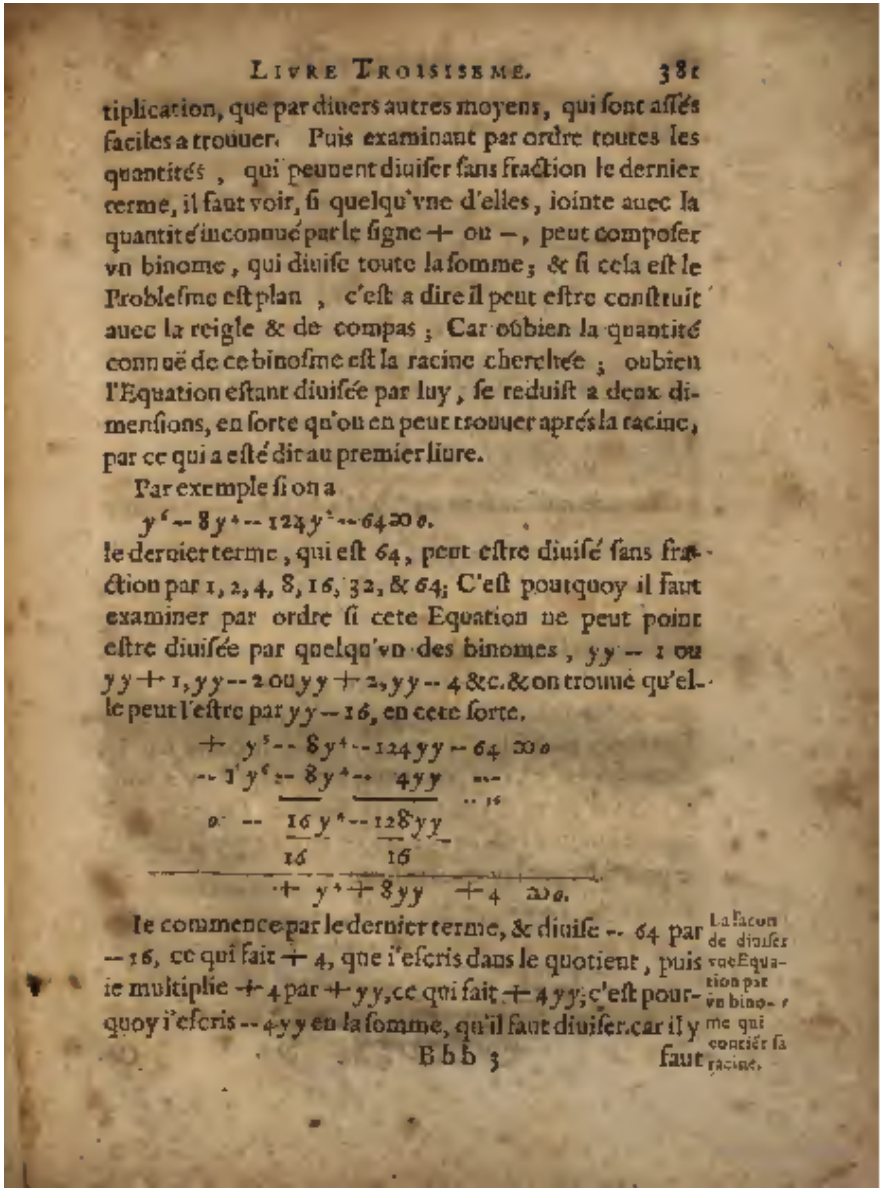
Polynomials feature in Descartes' *La géométrie* (1637), e.g.:

- one example to show that polynomials can be constructed from their roots (influenced by Harriot?);
- 'rule of signs': the number of positive ('true') roots of a polynomial is at most the number of times that the sign changes as we read term-by-term; the number of negative ('false') roots is at most the number of successions of the same sign; for example,

$$x^4 - 4x^3 - 19xx + 106x - 120 = 0$$

  has at most 3 positive roots and at most one negative;
- can always make a transformation to remove the second-highest term.

Descartes on cubics



LIVRE TROISIESME. 381

tiplication, que par diuers autres moyens, qui ſont aſſés faciles a trouuer. Puis examinant par ordre toutes les quantités, qui peuuent diuiſer ſans fraction le dernier terme, il faut voir, ſi quelqu'vne d'elles, iointe auec la quantité inconnuë par le ſigne + ou −, peut compoſer vn binome, qui diuiſe toute la ſomme; & ſi cela eſt le Probleſme eſt plan, c'eſt a dire il peut eſtre conſtruit auec la reigle & de compas; Car oubien la quantité connuë de ce binoſme eſt la racine cherchée; oubien l'Equation eſtant diuiſée par luy, ſe reduiſt a deux dimenſions, en ſorte qu'on en peut trouuer aprés la racine, par ce qui a eſté dit au premier liure.

Par exemple ſi on a

$y^6 - 8y^4 - 124y^2 - 64 \propto 0.$

le dernier terme, qui eſt 64, peut eſtre diuiſé ſans fraction par 1, 2, 4, 8, 16, 32, & 64; C'eſt pourquoy il faut examiner par ordre ſi cete Equation ne peut point eſtre diuiſée par quelqu'vn des binomes, $yy - 1$ ou $yy + 1$, $yy - 2$ ou $yy + 2$, $yy - 4$ &c. & on trouue qu'elle peut l'eſtre par $yy - 16$, en cete ſorte.

$$\begin{array}{rl} + & y^6 - 8y^4 - 124yy - 64 \propto 0 \\ - & y^6 - 8y^4 - 4yy \quad \text{\&c.} \\ \hline 0 & - 16y^4 - 128yy \\ & \quad 16 \qquad 16 \\ \hline & + y^4 + 8yy + 4 \propto 0. \end{array}$$

Ie commence par le dernier terme, & diuiſe − 64 par − 16, ce qui fait + 4, que i'eſcris dans le quotient, puis ie multiplie + 4 par + yy, ce qui fait + 4yy; c'eſt pourquoy i'eſcris − 4yy en la ſomme, qu'il faut diuiſer. car il y faut

La façon de diuiſer vne Equation par vn binome qui contiēt ſa racine.

Bbb 3

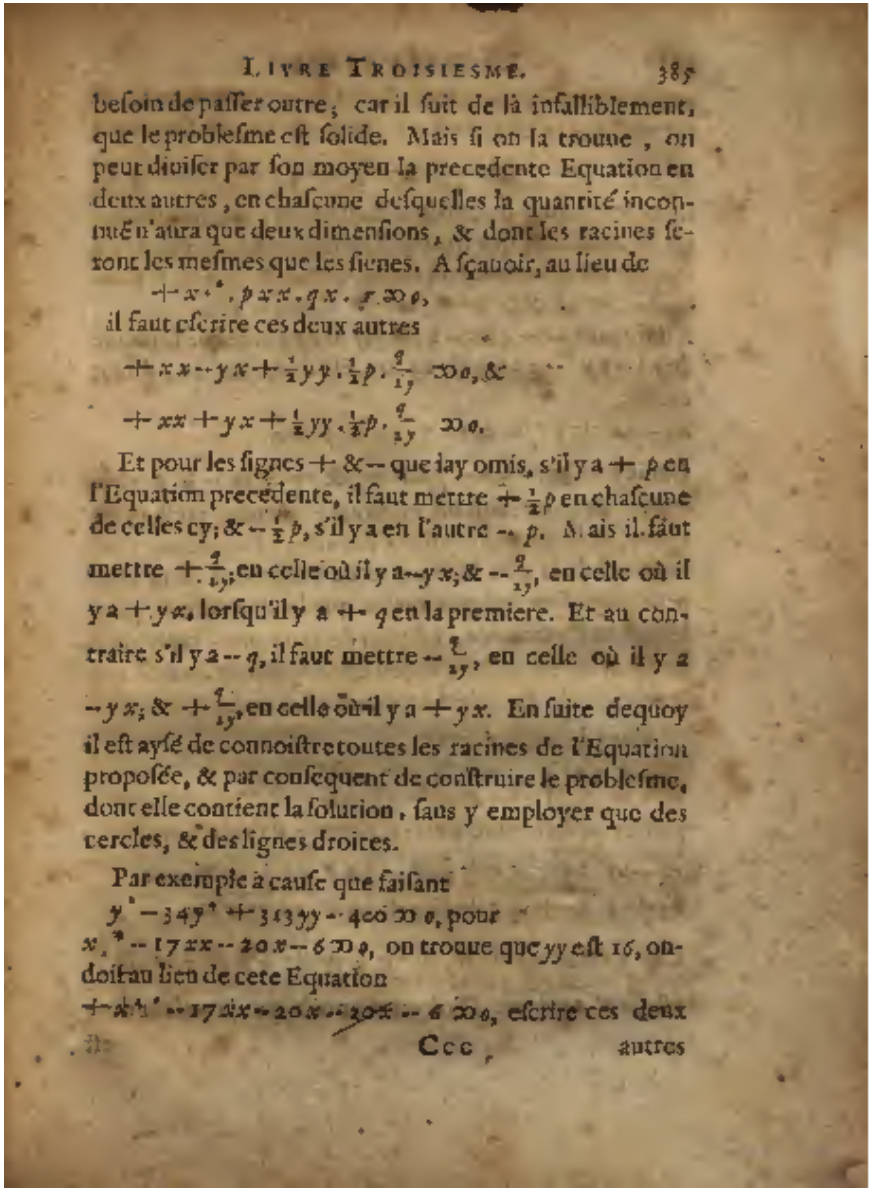
Search for roots of a cubic by examining the factors of the constant term:

if α is such a factor, test whether $x - \alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

Descartes on quartics



LIVRE TROISIESME. 385

besoin de passer outre; car il suit de là infalliblement, que le probleſme eſt ſolide. Mais ſi on la trouue, on peut diuiſer par ſon moyen la precedente Equation en deux autres, en chaſcune deſquelles la quantité inconnuë n'aura que deux dimenſions, & dont les racines ſeront les meſmes que les ſienes. A ſçauoir, au lieu de

$+x^4.pxx.qx.r \propto 0$,

il faut eſcrire ces deux autres

$+xx-yx+\frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \propto 0$, &

$+xx+yx+\frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \propto 0$.

Et pour les ſignes + & − que i'ay omis, s'il y a + p en l'Equation precedente, il faut mettre + $\frac{1}{2}p$ en chaſcune de celles cy; & − $\frac{1}{2}p$, s'il y a en l'autre − p. Mais il faut mettre + $\frac{q}{2y}$, en celle où il y a − yx; & − $\frac{q}{2y}$, en celle où il y a + yx, lorſqu'il y a + q en la premiere. Et au contraire s'il y a − q, il faut mettre − $\frac{q}{2y}$, en celle où il y a − yx; & + $\frac{q}{2y}$, en celle où il y a + yx. En ſuite dequoy il eſt ayſé de connoiſtre toutes les racines de l'Equation propoſée, & par conſequent de conſtruire le probleſme, dont elle contient la ſolution, ſans y employer que des cercles, & des lignes droites.

Par exemple à cauſe que faiſant

$y^6-34y^4+343yy-400 \propto 0$, pour

$x^4-17xx-20x-6 \propto 0$, on trouue que yy eſt 16, on doit au lieu de cete Equation

$+x^4-17xx-20x-30x-6 \propto 0$, eſcrire ces deux

To solve $+x^4 \star .pxx.qx.r = 0$ (Descartes' notation), that is,

$$x^4 \pm pxx \pm qx \pm r = 0\,,$$

he sought to write the quartic as a product of two quadratics. This led him to

$$y^6 \pm 2py^4 + (pp \pm 4r)yy - qq = 0$$

As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via $+$, $-$, $\times$, $\div$, $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, $\ldots$

NB: A solution in radicals may be constructed by ruler and compass.

Spoiler: the general quintic equation is not solvable in radicals.

By the 1770s, mathematicians (notably Lagrange) had come to suspect this, but it was not proved until the 1820s.

So did anything interesting happen in algebra during the 17th and 18th centuries?

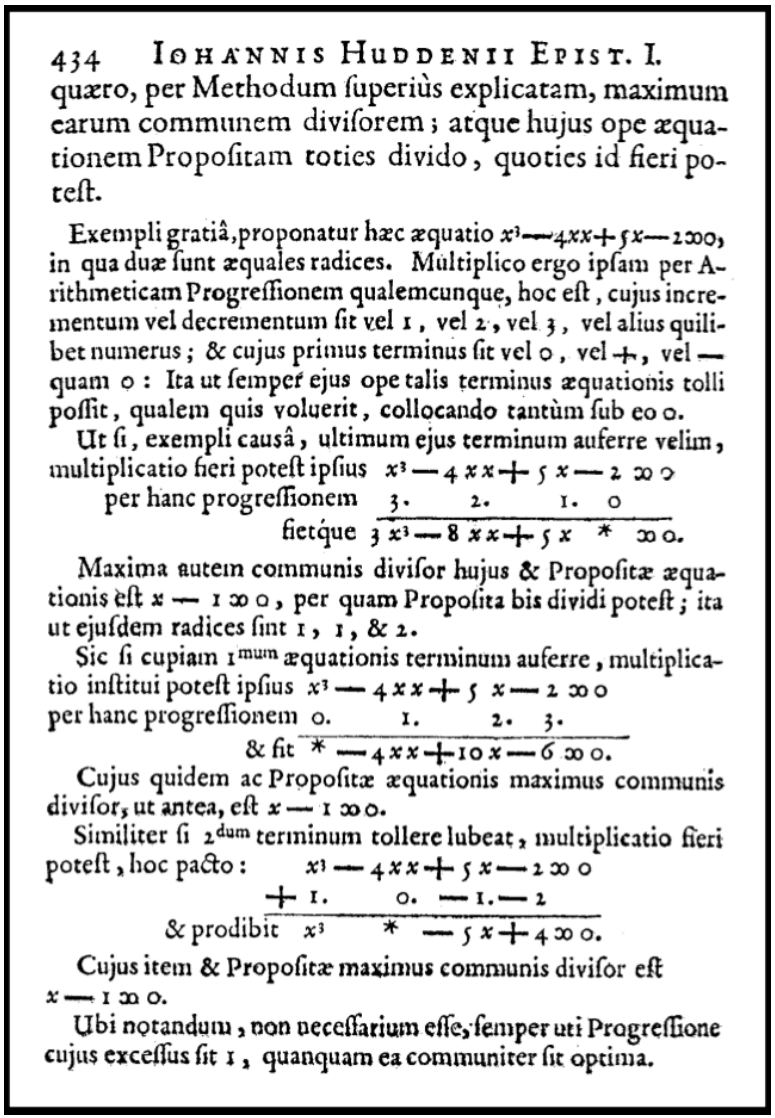
A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Fair point? Or not?

Some 17th-century developments: Hudde's rule (1657)



434 IOHANNIS HUDDENII EPIST. I.

quæro, per Methodum ſuperiùs explicatam, maximum earum communem diviſorem; atque hujus ope æquationem Propoſitam toties divido, quoties id fieri poteſt.

Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \propto 0$, in qua duæ ſunt æquales radices. Multiplico ergo ipſam per Arithmeticam Progreſſionem qualemcunque, hoc eſt, cujus incrementum vel decrementum ſit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus ſit vel 0, vel +, vel − quam 0: Ita ut ſemper ejus ope talis terminus æquationis tolli poſſit, qualem quis voluerit, collocando tantùm ſub eo 0.

Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \propto 0$
per hanc progreſſionem 3. 2. 1. 0
fietque $3x^3 - 8xx + 5x * \propto 0$.

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \propto 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1^mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius $x^3 - 4xx + 5x - 2 \propto 0$
per hanc progreſſionem 0. 1. 2. 3.
& fit $* - 4xx + 10x - 6 \propto 0$.

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \propto 0$.

Similiter ſi 2^dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \propto 0$
$+1. \quad 0. \quad -1. \quad -2$
& prodibit $x^3 \quad * \quad -5x + 4 \propto 0$.

Cujus item & Propoſitæ maximus communis diviſor eſt $x - 1 \propto 0$.

Ubi notandum, non neceſſarium eſſe, ſemper uti Progreſſione cujus exceſſus ſit 1, quanquam ea communiter ſit optima.

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:

$x^3 - 4xx + 5x - 2 = 0$ has a double root $x = 1$;

multiply the terms of the equation by numbers in arithmetic progression:

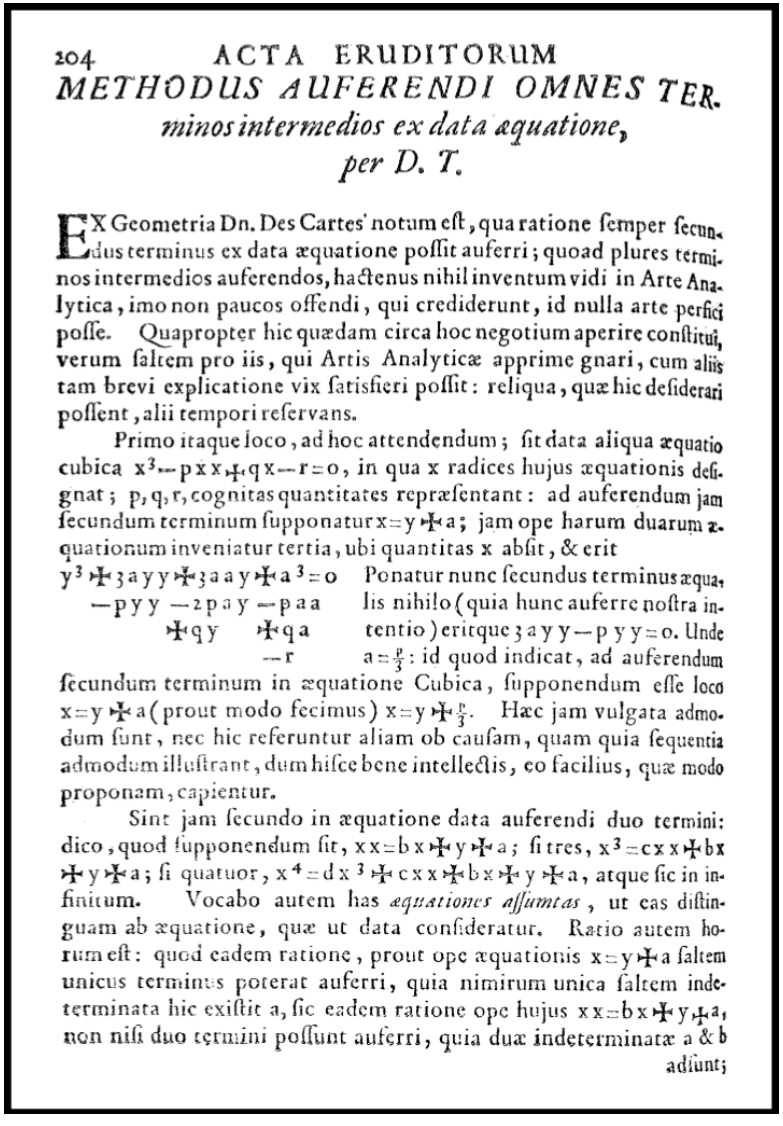
$3x^3 - 8xx + 5x = 0$ also has a double root $x = 1$,

as does $-4xx + 10x - 6 = 0$.

(Modern form of rule: if r is a double root of $f(x) = 0$, then it is a root of $f'(x) = 0$ also.)

See *Mathematics emerging*, §12.2.2.

Some 17th-century developments: Tschirnhaus transformations (1683)



204 ACTA ERUDITORUM

METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione, per D. T.

EX Geometria Dn. Des Cartes' notum eſt, qua ratione ſemper ſecundus terminus ex data æquatione poſſit auferri; quoad plures terminos intermedios auferendos, hactenus nihil inventum vidi in Arte Analytica, imo non paucos offendi, qui crediderunt, id nulla arte perfici poſſe. Quapropter hic quædam circa hoc negotium aperire conſtitui, verum ſaltem pro iis, qui Artis Analyticæ apprime gnari, cum aliis tam brevi explicatione vix ſatisfieri poſſit: reliqua, quæ hic deſiderari poſſent, alii tempori reſervans.

Primo itaque loco, ad hoc attendendum; ſit data aliqua æquatio cubica $x^3 - pxx + qx - r = o$, in qua x radices hujus æquationis deſignat; p, q, r, cognitas quantitates repræſentant: ad auferendum jam ſecundum terminum ſupponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas x abſit, & erit

$y^3 + 3ayy + 3aay + a^3 = o$
$-pyy - 2pay - paa$
$+qy + qa$
$-r$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = o$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

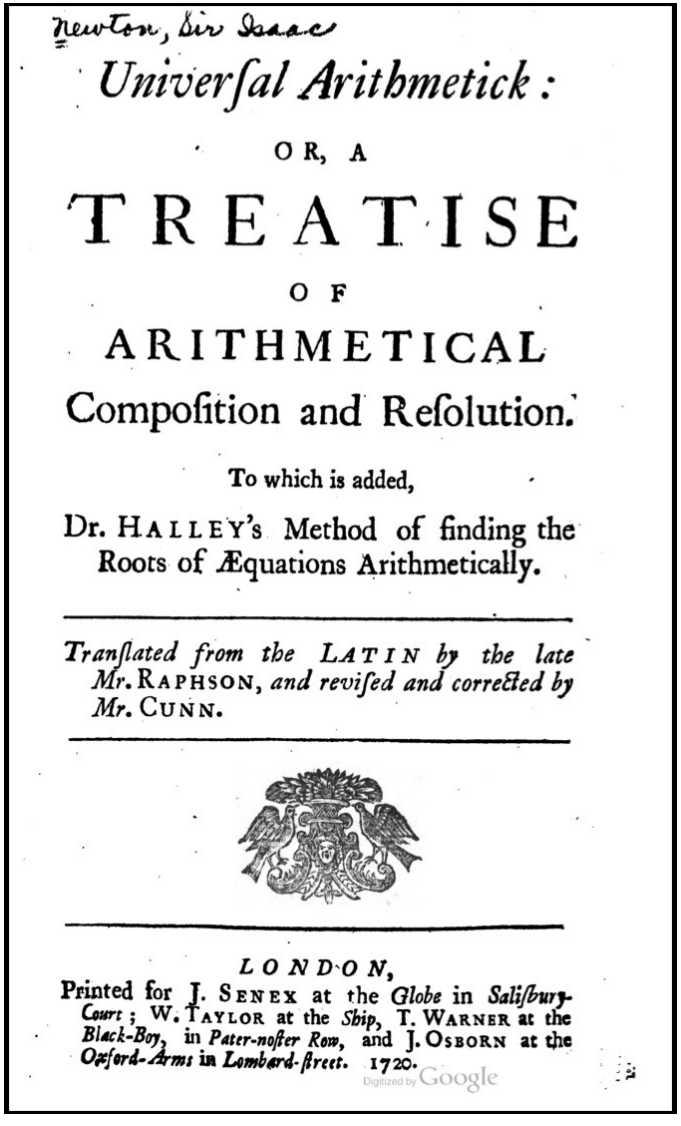
Sint jam ſecundo in æquatione data auferendi duo termini: dico, quod ſupponendum ſit, $xx = bx + y + a$; ſi tres, $x^3 = cxx + bx + y + a$; ſi quatuor, $x^4 = dx^3 + cxx + bx + y + a$, atque ſic in infinitum. Vocabo autem has *æquationes aſſumtas*, ut eas diſtinguam ab æquatione, quæ ut data conſideratur. Ratio autem horum eſt: quod eadem ratione, prout ope æquationis $x = y + a$ ſaltem unicus terminus poterat auferri, quia nimirum unica ſaltem indeterminata hic exiſtit a, ſic eadem ratione ope hujus $xx = bx + y + a$, non niſi duo termini poſſunt auferri, quia duæ indeterminatæ a & b adſunt;

For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?
- to remove two terms put $x^2 = bx + y + a$

See *Mathematics emerging*, §12.2.3.

An 18th-century development: Newton's *Arithmetica universalis* (1707)



Newton, Sir Isaac

Universal Arithmetick:

OR, A

TREATISE

OF

ARITHMETICAL

Composition and Resolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

Translated from the LATIN by the late Mr. RAPHSON, *and revised and corrected by Mr.* CUNN.

LONDON,

Printed for J. SENEX at the *Globe* in *Salisbury-Court*; W. TAYLOR at the *Ship*, T. WARNER at the *Black-Boy*, in *Pater-noster Row*, and J. OSBORN at the *Oxford-Arms* in *Lombard-street*. 1720.

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Rules for sums of powers of roots of

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \cdots = 0$$

$$\begin{aligned} \text{sum of roots} &= p \\ \text{sum of roots}^2 &= pa - 2q \\ \text{sum of roots}^3 &= pb - qa + 3r \\ \text{sum of roots}^4 &= pc - qb + ra - 4s \end{aligned}$$

Developments of the 17th and 18th centuries

- Symbolic notation
- Understanding of the structure of polynomials
- . . . of the number and nature of their roots
- . . . of the relationship between roots and coefficients
- . . . of how to manipulate them
- . . . of how to solve them numerically
- The leaving behind of geometric intuition?

Some 18th-century theory of equations

Recall:

- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

Some 18th-century theory of equations

Recall:

- quadratic equations can be solved by means of linear equations
- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

The ‘reduced’ or ‘resolvent’ equation:

- for cubics, the reduced equation is of degree 2
- for quartics, the reduced equation is of degree 3
- for quintics, the reduced equation is of degree ?

Some 18th-century hypotheses

Euler's hypothesis (1733):

- for an equation of degree n the degree of the reduced equation will be $n - 1$

Bézout's hypothesis (1764):

- for an equation of degree n the degree of the reduced equation will in general be $n!$
- though always reducible to $(n - 1)!$
- possibly further reducible to $(n - 2)!$

Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

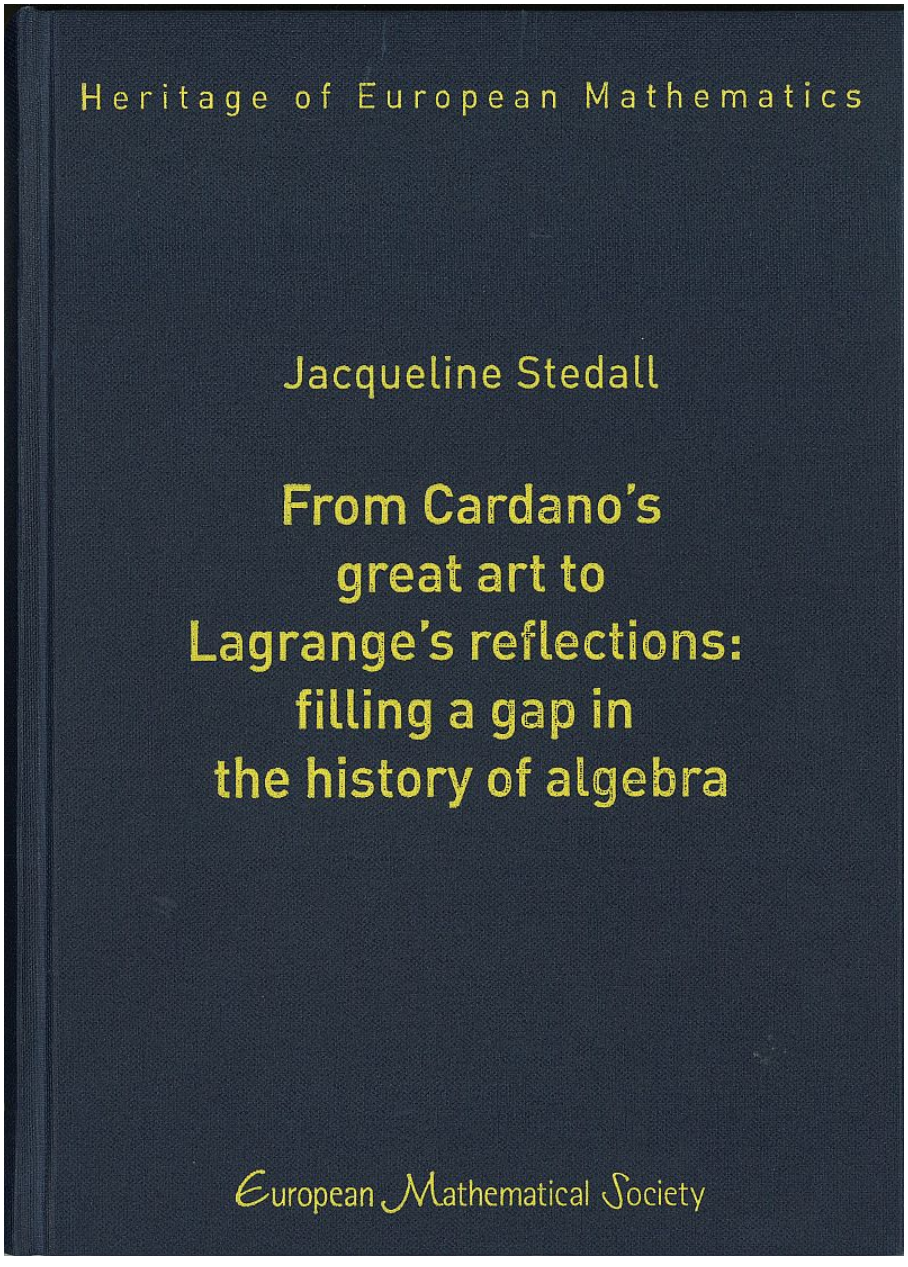
seeking to identify a uniform method that could be extended to higher degree

A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Filling a gap in the history of algebra (2011)



The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations . . . [Preface]

From Stedall's preface:

This assertion ... from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'. Fortunately, the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.