Matrix eigenvalue problems and linear systems

Introduction

Eigenvalue problems $Ax = \lambda x$ and linear systems $Ax = b$ are the two central problems addressed in numerical linear algebra. These are both of fundamental importance in scientific computing, as most problems eventually boil down to one of the two (or their variant), taking up a bulk of the computational effort.

The Singular Value Decomposition (SVD) $A = U\Sigma V^T$, which is closely related to eigenvalue problems, is also worth the special mention for its enormous importance in applications in data science.

While extensive studies have been devoted to these problems, a number of open problems remain, and many interesting questions have been identified with the rapid surge of data science, with significant ramifications in applications. For example, the randomised Kaczmarz method is now an attractive method for linear systems when the matrix is too large to fit in memory and one is willing to settle with a good approximate solution rather than an exact solution; but its convergence behavior is not fully understood. Other topics recently developed and awaiting further investigations include the Cholesky QR algorithm for QR factorisation, and genearlisations of Sylvester's law of inertia. There is also room for further explorations in classical topics in matrix analysis and eigenvalue perturbation theory.

Project

This project aims to explore, experiment, examine and potentially resolve problems in numerical linear algebra. Possible topics include

1. Explorations and convergence analysis of the randomised Kaczmarz method
2. Explorations of the Cholesky QR algorithm, e.g. for spectral divide-and-conquer methods for eigenvalue problems and the SVD
3. Investigate analyticity of eigenvalues and eigenvectors near crossing (requires complex analysis background)
4. Generalisations of Sylvester's law of inertia
5. Matrix (eigenvalue/vector) perturbation theory

Prerequisites

Prelims Linear Algebra. Part A Numerical Analysis is highly recommended. Part A courses on Probability and Complex Analysis would also be helpful.

Reading

The general recommended textbook is: L. N. Trefethen and D. Bau, Numerical Linear Algebra, SIAM, 1997.

For specific topics,

T. Fukaya, R. Kannan, Y. Nakatsukasa, Y. Yamamoto, and Y. Yanagisawa. Shifted cholesky QR for computing the qr factorization of ill- conditioned matrices. SIAM J. Sci. Comp, 42(1):A477–A503, 2020. (for 2)

Y. Nakatsukasa. Sharp error bounds for ritz vectors and approximate singular vectors. Math. Comp., 89(324):1843–1866, 2020. (for 5)

Y. Nakatsukasa and V. Noferini. Inertia laws and localization of real eigenvalues for generalized indefinite eigenvalue problems. Linear Algebra Appl., 578:272296, 2019. (for 4)

G. W. Stewart and J.-G. Sun. Matrix Perturbation Theory. Academic Press, 1990. (for 5)

T. Strohmer and R. Vershynin. A randomized Kaczmarz algorithm with exponential convergence. J. Fourier Anal. Appl., 15(2):262, 2009. (for 1)

L. N. Trefethen, Chebfun example `https://www.chebfun.org/examples/linalg/CrossingsAnalyticity.html` (for 3)