B8.5 Graph Theory **Problem Sheet 1** **17/10/2019**

1. Prove that if a graph G has at least two vertices then G contains two vertices of the same degree.

2. (a) Find all connected graphs G that do not contain the complete bipartite graph $K_{1,2}$ as a subgraph.

   (b) Find all connected graphs G that do not contain $K_{1,2}$ as an *induced* subgraph.

3. Find all connected graphs G with $\Delta(G) \leqslant 2$.

4. A graph is *r-regular* if every vertex has degree exactly r. Prove that there is a 3-regular graph on n vertices if and only if n is even and $n \geqslant 4$.

5. Show that if T is a tree that is not a path, then T has at least three leaves. Can you classify all trees with exactly three leaves?

6. Let $f(n)$ be the number of isomorphism classes of connected n-vertex graphs in which every vertex has degree at most 3. Show that $f(n) \to \infty$. Show further that there are constants $A, c > 0$ such that $f(n) \geqslant Ae^{cn}$ for every n.

7. Find (draw) the tree on $[9]$ with Prüfer code 6423743.

8. Consider the algorithm in the lecture notes mapping a Prüfer code $\mathbf{c} = (c_1, c_2, \ldots, c_{n-2})$ to a tree T on $[n]$. Show that T has code $\mathbf{c}$.

9. Let $k \geqslant 1$, and suppose that G is a connected $2k$-regular graph.

   (a) Prove that if G has an even number of edges then there is a k-regular subgraph H of G such that $V(H) = V(G)$. [Hint: G has an Euler circuit.]

   (b) What can you say if G has an odd number of edges?

10. The *discrete cube* Q_n has vertex set $\{0,1\}^n$, and two vertices are joined if they differ in exactly one coordinate. (Thus $|Q_n| = 2^n$ and $e(Q_n) = n2^{n-1}$.) Prove that Q_n contains a Hamilton cycle for every $n \geqslant 2$.

11. For each integer $k \geqslant 1$, find a connected, *non-complete* graph G containing no P_{2k+1} with $\bar{d}(G) \geqslant 2k - 0.0001$. (Hint: try $k = 1$ first.)

Optional bonus questions. These may not be covered in classes; MFOCS students should attempt them!

12. For $1 \leqslant k \leqslant n$ a graph G on $[n]$ is a $[k]$-*forest* if it is acyclic and has exactly k components, with the vertices $1, 2, \ldots, k$ in distinct components. Let $a_{n,k}$ be the number of $[k]$-forests on $[n]$, and set $a_{n,0} = 0$. Show that

$$a_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} a_{n-1,k-1+i}$$

for any $n \geqslant 2$ and $1 \leqslant k \leqslant n$, and hence that $a_{n,k} = kn^{n-k-1}$. Deduce Cayley's formula.

[Hint: recall the Binomial Theorem, and also that $\binom{a}{b} = \frac{a}{b}\binom{a-1}{b-1}$ for $a \geqslant b \geqslant 1$.]

13. The *average degree* of $G = (V, E)$ is $|V|^{-1} \sum_{v \in V} d(v)$. Let $G = (V, E)$ be a graph with average degree d and without isolated vertices.

(a) Show that there is a vertex $v \in V$ so that the average degree of the neighbours of v is at least d.

(b) Must there be a vertex $v \in V$ so that the average degree of the neighbours of v is at most d?

14. Suppose that G and H are infinite graphs, and that G is isomorphic to a subgraph of H and H is isomorphic to a subgraph of G. Must G and H be isomorphic?

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk