B8.5 Graph Theory **Problem Sheet 2** **30/10/2019**

1. Find graphs G_1 and G_2 overlapping in two vertices which are not adjacent in either graph such that $\chi(G_1 \cup G_2) > \max(\chi(G_1), \chi(G_2))$. How big can the difference be?

2. Let G be a connected graph with n vertices and let v be a given vertex of G. Show that we can list the vertices as $v_1, v_2, \ldots, v_{n-1}, v_n = v$ so that each v_i, $i < n$, has at least one later neighbour, i.e., at least one neighbour v_j with $j > i$.

3. Show that $e(G) \geqslant \binom{\chi(G)}{2}$ for every graph G.

4. Let G be a graph of order n, and write $\overline{G}$ for the complement of G. Prove that $\chi(G)\chi(\overline{G}) \geqslant n$ and deduce that $\chi(G) + \chi(\overline{G}) \geqslant 2\sqrt{n}$.

5. Suppose that $\chi(G) = k$ and $c : V(G) \to \{1, \ldots, k\}$ is a proper k-colouring of G. Must there be a path $x_1 \cdots x_k$ in G with $c(x_i) = i$ for each i?

6. (a) Show that for every graph G there is an ordering of $V(G)$ for which the greedy algorithm uses $\chi(G)$ colours.

   (b) Find a bipartite graph and an ordering of its vertices so that the greedy algorithm uses 2019 colours.

7. (a) Prove that if T is a tree with n vertices then its chromatic polynomial is $p_T(x) = x(x-1)^{n-1}$.

   (b) Show that the chromatic polynomial of C_n is given by $p_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$ for each $n \geqslant 3$.

   (c) For $n \geqslant 3$, the *wheel* W_{n+1} is the graph with $n+1$ vertices obtained from C_n by adding a new vertex adjacent to everything. Calculate the chromatic polynomial of W_{n+1}.

8. Let G be a graph of order $n \geqslant 3$ and let $p_G(x) = \sum_{i=0}^{n-1}(-1)^i a_i x^{n-i}$ be the chromatic polynomial of G. Recall that $a_0 = 1$, $a_1 = e(G)$. Show that $a_2 = \binom{e(G)}{2} - t(G)$, where $t(G)$ is the number of copies of K_3 in G.

9. Find graphs G and H such that $|G| = |H|$, $e(G) = e(H)$, $\chi(G) > \chi(H)$ and $p_G(x) > p_H(x)$ for all sufficiently large x.

10. (a) Show that if G has the same chromatic polynomial as K_n then $G \cong K_n$.

    (b) Show that if G has the same chromatic polynomial as $K_{n,n}$ then $G \cong K_{n,n}$.

Optional bonus questions. These may not be covered in classes; MFOCS students should attempt them!

11. For which k and ℓ can you construct a graph G with $\chi(G) = k$ and $\omega(G) = \ell$?

12. An *acyclic orientation* of G is a way of assigning a direction $u \to v$ or $v \to u$ to each edge uv of G so that there is no cycle $v_1 \to v_2 \to \cdots v_t \to v_1$. Show that $|p_G(-1)| = (-1)^{|G|} p_G(-1)$ is the number of acyclic orientations of G.

13. Let G be the infinite graph with vertex set $\mathbb{R}^2$ in which two vertices are joined if and only if they are at Euclidean distance 1. Prove that $4 \leqslant \chi(G) \leqslant 7$. [Remark: these were the best known bounds for many years. But in 2018 Aubrey de Grey showed that $\chi(G) \geqslant 5$.]

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk