1. We are given a deck of $n$ cards in order $1,2,\ldots,n$. Then a randomly chosen card is removed and placed at a random position in the deck. What is the entropy of the resulting deck of cards?

2. **(Pooling inequalities)** Let $a,b \geq 0$ with $a+b>0$. Show that
$$-(a+b)\log(a+b) \leq -a\log a - b\log b \leq -(a+b)\log\frac{a+b}{2}$$
and that the first inequality becomes an equality iff $ab=0$, the second inequality becomes an equality iff $a=b$.

3. **(Log sum inequality)** Let $a_1,\ldots,a_n,b_1,\ldots,b_n \geq 0$. Show that
$$\sum_{i=1}^n a_i \log\frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i\right)\log\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$$
with equality iff $\frac{a_i}{b_i}$ is constant.

4. Let $X,Y,Z$ be discrete random variables. Prove or provide a counterexample to the following statements:
   (a) $H(X) = H(-42X)$,
   (b) $H(X|Y) \geq H(X|Y,Z)$,
   (c) $H(X,Y) = H(X)+H(Y)$.

5. Does there exist a discrete random variable $X$ with a distribution such that $H(X)=\infty$? If so, describe it as explicitly as possible.

6. Let $\mathcal{X}$ be a finite set, $f$ a real-valued function $f:\mathcal{X}\to\mathbb{R}$ and fix $\alpha\in\mathbb{R}$. We want to maximise the entropy $H(X)$ of a random variable $X$ taking values in $\mathcal{X}$ subject to the constraint
$$\mathbb{E}[f(X)] \leq \alpha. \tag{1}$$
Therefore show that if $U$ denotes a uniformly distributed random variable on $\mathcal{X}$, it holds that
   (a) if $\max_{x\in\mathcal{X}} f(x) \geq \alpha \geq \mathbb{E}[f(U)]$, then the entropy is maximised subject to (1) by the uniformly distributed random variable $U$.
   (b) if $f$ is non-constant and $\min_{x\in\mathcal{X}} f(x) \leq \alpha < \mathbb{E}[f(U)]$, then the entropy is maximised subject to (1) by the random variable $Z$ given by
$$\mathbb{P}(Z=x) = \frac{\exp(\lambda f(x))}{\sum_{x'\in\mathcal{X}}\exp(\lambda f(x'))} \text{ for } x\in\mathcal{X}$$
   where $\lambda<0$ is chosen such that $\mathbb{E}[f(Z)]=\alpha$.
   (c) * **(Optional)** Prove that under the assumptions of (b) the choice for $\lambda$ is unique and we have $\lambda<0$.

7. * **(Optional)** Partition the interval $[0,1]$ into $n$ disjoint sub-intervals of length $p_1,\ldots,p_n$. Let $X_1,X_2,\ldots$ be iid random variables, uniformly distributed on $[0,1]$, and $Z_m(i)$ be the number of the $X_1,\ldots,X_m$ that lie in the $i$th interval of the partition. Show that the random variables
$$R_m = \prod_{i=1}^n p_i^{Z_m(i)} \text{ satisfy } \frac{1}{m}\log R_m \xrightarrow{m\to\infty} \sum_{i=1}^n p_i\log p_i \text{ with probability 1.}$$

8. * **(Optional, revision: probability theory)** Let $X$ be a real-valued random variable.

(a) Assume additionally that $X$ is non-negative. Show that for every $x > 0$ we have

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}[X]}{x}.$$

(b) Let $X$ be a random variable of mean $\mu$ and variance $\sigma^2$. Show that $\mathbb{P}(|X - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$.

(c) Let $(X_n)_{n\geq 1}$ be a sequence of identically distributed, independent random variables with mean $\mu$ and variance $\sigma^2$. Show that for every $\epsilon > 0$

$$\lim_{m\to\infty} \mathbb{P}\left(\left|\frac{1}{m}\sum_{n=1}^{m} X_n - \mu\right| > \epsilon\right) = 0$$

(d) Let $X$ be uniformly distributed on $\left[0, \frac{\pi}{2}\right]$. Find the density of $Y = \sin X$.