2019

2. Let $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ be a complete filtered probability space, where $(\mathcal{F}_n)_{n\geqslant 0}$ is a filtration, i.e. an increasing sequence of sub σ-algebras of $\mathcal{F}$.

(a) [5 marks]

(i) What does it mean that $X = (X_n)_{n\geqslant 0}$ is a martingale? What does it mean that $X = (X_n)_{n\geqslant 0}$ is a sub-martingale?

(ii) State Doob's martingale convergence theorem.

(iii) If $X = (X_n)_{n\geqslant 0}$ is a martingale, show that $X^+ = (X_n^+)_{n\geqslant 1}$ is a sub-martingale, where $X_n^+ = \max\{X_n, 0\}$ for every $n \geqslant 0$.

(b) [15 marks] Let $X = (X_n)_{n\geqslant 0}$ be a martingale with $X_0 = 0$. Define $\xi_n = X_n - X_{n-1}$ (for $n = 1, 2, \cdots$), and

$$A = \left\{ \sup_n X_n < \infty \text{ and } \inf_n X_n > -\infty \right\}.$$

Suppose that there is a positive constant L such that $|\xi_n| \leqslant L$ for every $n \geqslant 1$.

(i) For a positive number a, let $T_a = \inf\{n \geqslant 0 : X_n > a\}$. Show that T_a is a stopping time, and show that the stopped random sequence $(X_{T_a\wedge n})_{n\geqslant 0}$ is a martingale, and $\mathbb{E}[X_{T_a\wedge n}] = 0$ for all $n \geqslant 0$. Deduce that

$$\mathbb{E}[|X_{T_a\wedge n}|] = 2\mathbb{E}\left[X_{T_a\wedge n}^+\right]$$

for every $a > 0$ and every $n \geqslant 0$. Here, for a stopping time T and a real random sequence $(Y_n)_{n\geqslant 0}$ you might find it useful to recall that

$$Y_{T\wedge n}^+ = \sum_{k=0}^{n} Y_k^+ 1_{\{T=k\}} + Y_n^+ 1_{\{T>n\}}$$

for every $n \geqslant 0$. Here, we recall that $Y_{T\wedge n} = Y_T$ on $T \leqslant n$ and $Y_{T\wedge n} = Y_n$ on $T > n$ by definition, and similar notation applies to Y^+.

(ii) Show that

$$X_{T_a\wedge n}^+ \leqslant X_{T_a\wedge(n-1)}^+ + \xi_{T_a\wedge n}^+$$

for every $n \geqslant 1$. Hence, or otherwise show that

$$X_{T_a\wedge n}^+ \leqslant a + L$$

for every $a > 0$ and for every $n \geqslant 1$.

(iii) Show that, for every $a > 0$, $\left(X_{T_a\wedge n}^+\right)_{n\geqslant 0}$ converges almost everywhere. Hence or otherwise, show that $(X_n^+)_{n\geqslant 0}$ converges almost everywhere on $\{\sup_n X_n \leqslant a\}$ for every $a > 0$. Deduce that $(X_n^+)_{n\geqslant 0}$ converges almost everywhere on $\{\sup_n X_n < \infty\}$, and hence show that $(X_n)_{n\geqslant 0}$ converges almost everywhere on A.

(c) [5 marks] Let $(Z_n)_{n\geqslant 1}$ be an adapted random sequence, i.e. Z_n is $\mathcal{F}_n$-measurable for every $n \geqslant 1$. Suppose $0 \leqslant Z_n \leqslant 1$ for every $n \geqslant 1$. Define $Y_0 = 0$ and

$$Y_n = Y_{n-1} + Z_n - \mathbb{E}[Z_n|\mathcal{F}_{n-1}]$$

for $n \geqslant 1$. Show that $(Y_n)_{n\geqslant 0}$ is a martingale with $\mathbb{E}[Y_n] = 0$, and show that Y_n converges almost everywhere on

$$\left\{ \sup_n Y_n < \infty \text{ and } \inf_n Y_n > -\infty \right\}.$$

2018 2. Let $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space.

(a) [5 marks] Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and X be integrable. What does it mean to say that $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$? Show that the conditional expectation of X given $\mathcal{G}$ is unique up to almost surely. Give a definition of $\mathbb{E}[X|Z]$ where Z is also a random variable.

(b) [4 marks] Suppose X and Y are two independent, square integrable random variables with the same distribution.
What are $\mathbb{E}[X|X+Y]$ and $\mathbb{E}\left[X^2+XY|X+Y\right]$?

(c) [6 marks] What does it mean that a random sequence $(H_n)_{n\geqslant 0}$ is predictable? Give a definition of a stopping time.
Suppose now $M=(M_n)_{n\geqslant 0}$ is a martingale and $H=(H_n)_{n\geqslant 0}$ is predictable and bounded. Define $H.M$ to be the sequence given inductively by the equations that $(H.M)_0=0$ and

$$(H.M)_n = (H.M)_{n-1} + H_n\left(M_n - M_{n-1}\right)$$

for $n\geqslant 1$. Show that $H.M$ is a martingale.
Let T be a stopping time, and $H_n = 1_{\{T\geqslant n\}}$ for $n=0,1,2,\cdots$. Show that $(H_n)_{n\geqslant 0}$ is predictable and $(H.M)_n = M_{T\wedge n} - M_0$ for every n. Hence, or otherwise, show that $(M_{T\wedge n})_{n\geqslant 0}$ is a martingale.

(d) [3 marks] Suppose that $(M_n)_{n\geqslant 0}$ is a non-negative martingale, and T is a finite stopping time. Show that $\mathbb{E}[M_T]\leqslant\mathbb{E}[M_0]$. Give a sufficient condition on T, such that $\mathbb{E}[M_T]=\mathbb{E}[M_0]$.

(e) [7 marks] State Doob's martingale convergence theorem.
Let $(\xi_n)_{n\geqslant 1}$ be a sequence of independent identically distributed random variables with

$$\mathbb{P}\left[\xi_1=\frac{1}{2}\right]=\mathbb{P}\left[\xi_1=\frac{3}{2}\right]=\frac{1}{2}.$$

Let $X_n=\xi_1\cdots\xi_n$ and $X_0=1$. Show that $(X_n)_{n\geqslant 0}$ is a martingale, and X_n converges to X_∞ as $n\to\infty$ with probability one. By using the Strong Law of Large Numbers for some sequence, or otherwise, further show that $X_\infty=0$. Hence conclude that $\mathbb{E}(X_\infty)\neq\mathbb{E}(X_n)$ for every n.

2017

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, so that $(\Omega, \mathcal{F})$ is a measurable space, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$.

   (a) [6 marks] Let $\mathcal{G}_n \subseteq \mathcal{F}$ (where n runs through an index set Λ) be a family of sub σ-algebras of $\mathcal{F}$. What does it mean that $\{\mathcal{G}_n : n \in \Lambda\}$ are *independent*?

   Suppose X is a real random variable on $(\Omega, \mathcal{F})$. Define the *σ-algebra $\sigma\{X\}$ generated by X*.

   Suppose $\{X_n : n \in \Lambda\}$ is a family of real random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. What does it mean that $\{X_n : n \in \Lambda\}$ are *independent*?

   (b) [3 marks] What does it mean that a collection $\mathcal{C}$ of some subsets of Ω is a *π-system*?

   State the *uniqueness lemma* for two finite measures on $(\Omega, \mathcal{F})$.

   (c) [8 marks] Suppose X, Y and Z are three independent real random variables. Prove that $\sin(X+Y)$ and Z are independent.

   [*Hint: You may first prove that $\sigma\{X, Y\}$ and $\sigma\{Z\}$ are independent, using the uniqueness lemma for finite measures or otherwise.*]

   (d) [8 marks] (i) Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra, and X be an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. What does it mean that $\mathbb{E}[X|\mathcal{G}]$ is the *conditional expectation* of X given $\mathcal{G}$? If X and Y are two integrable random variables, what is the definition of $\mathbb{E}[X|Y]$ the *conditional expectation* of X given Y?

   (ii) Suppose X and Y are independent, and

   $$\mathbb{P}[X=0] = \frac{1}{3} \text{ and } \mathbb{P}\left[X = \frac{\pi}{2}\right] = \frac{2}{3}.$$

   Find $\mathbb{E}[\sin(XY)|Y]$ and justify your answer.

2017

2. (a) [13 marks] Let $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space.

   (i) Let $M = (M_n)_{n\geqslant 0}$ be a sequence of random variables. What does it mean that (M_n) is a *super-martingale*?

   (ii) Suppose (X_n) and (Y_n) are two super-martingales. Prove that $(X_n \wedge Y_n)$ is a super-martingale, where $x \wedge y = \min\{x, y\}$.

   (iii) What does it mean that T is an $(\mathcal{F}_n)$-*stopping time*? What does it mean that a sequence of random variables $H = (H_n)$ is *predictable* with respect to $(\mathcal{F}_n)$?

   (iv) Suppose (M_n) is a sub-martingale and (H_n) is predictable, such that H_n is non-negative, and $H_n M_n$ and $H_n M_{n-1}$ are integrable for $n = 1, 2, \ldots$. Define $X_0 = 0$ and

$$X_n = \sum_{k=1}^{n} H_k \left(M_k - M_{k-1}\right)$$

   for $n = 1, 2, \ldots$. Show that (X_n) is also a sub-martingale.

(b) [12 marks] Let $\{\eta_n : n = 1, 2, \ldots\}$ be a sequence of independent random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distributions given by

$$\mathbb{P}\left[\eta_n = 1\right] = \mathbb{P}\left[\eta_n = -1\right] = \frac{1}{(n+1)^p}, \quad \text{and} \quad \mathbb{P}\left[\eta_n = 0\right] = 1 - \frac{2}{(n+1)^p},$$

where $p > 1$ is a constant. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \eta_k$ for $n \geqslant 1$.

   (i) Give a definition of the *tail σ-algebra* $\mathcal{G}_\infty$ for $\{\eta_n : n = 1, 2, \ldots\}$, and state *Kolmogorov's 0-1 law.* Show that

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > 1\right] = 0 \text{ or } 1.$$

   (ii) State the *Borel–Cantelli lemma* for a sequence of independent events. Let $A_n = \{|\eta_n| > 0\}$ for $n = 1, 2, \ldots$. Show that $\mathbb{P}\left[A_n \text{ i. o. }\right] = 0$, and hence or otherwise prove that $\eta_n \to 0$ with probability one, where $\{A_n \text{ i. o. }\}$ denotes the event that infinitely many A_n occur.

2016

2. (a) [15 marks] Let $(X_n)_{n\geqslant 1}$ be a sequence of independent real random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (i) Define the *tail σ-algebra* $\mathcal{G}_\infty$ with respect to $(X_n)_{n\geqslant 1}$, and state *Kolmogorov's zero-one law* for the events in $\mathcal{G}_\infty$.

   (ii) Show that both $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ are $\mathcal{G}_\infty$-measurable, and deduce that $\{\omega \in \Omega : \limsup_{n\to\infty} X_n(\omega) \in G\}$ is $\mathcal{G}_\infty$-measurable for every Borel measurable subset $G \subseteq \mathbb{R}$. Hence show that $\limsup_{n\to\infty} X_n$ equals a constant almost surely.

   (iii) Show that $\mathbb{P}[\lim_{n\to\infty} X_n = 0] = 1$ if and only if

$$\sum_{n=1}^{\infty} \mathbb{P}[|X_n| > \varepsilon] < \infty$$

   for every $\varepsilon > 0$.
   [*The Borel–Cantelli Lemma may be used as long as you state it clearly.*]

(b) [10 marks] Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra, and let X be an integrable variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

   (i) What does it mean that $\mathbb{E}[X|\mathcal{G}]$ is the *conditional expectation of X given $\mathcal{G}$*?

   (ii) If Z is another random variable, what does it mean that $\mathbb{E}[X|Z]$ is the *conditional expectation of X given Z*? Suppose X, Y and Z are three independent, integrable random variables with the same distribution. What is $\mathbb{E}[X|X+Y+Z]$?

   (iii) Suppose X and Y are square integrable random variables, such that $\mathbb{E}[X|Y] = Y$ and $\mathbb{E}[Y|X] = X$. Show that $X = Y$ almost surely.

2015 1. (a) [7 marks] Define the terms *σ-algebra*, *π-system*, *measure* and *probability space*. What is the *σ-algebra generated by* a collection $\mathcal{A}$ of subsets of a given set Ω?

State Carathéodory's Extension Theorem giving conditions under which a set function μ_0 on $\mathcal{A} \subseteq \mathcal{F}$ extends to a measure on $(\Omega, \mathcal{F})$, and a corresponding result giving conditions under which such an extension is necessarily unique.

(b) [6 marks] Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 1, 2$, be two probability spaces. Define the *product σ-algebra* $\mathcal{F}_1 \times \mathcal{F}_2$. Let $\mathcal{R} = \{A_1 \times A_2 : A_i \in \mathcal{F}_i\}$ and let $\mathcal{A}$ be the set of finite disjoint unions of elements of $\mathcal{R}$, which you may assume is an algebra. Define a set function μ on $\mathcal{A}$ by $\mu(A_1 \times A_2) = \mathbb{P}_1[A_1]\mathbb{P}_2[A_2]$ for $A_i \in \mathcal{F}_i$ and $\mu(R_1 \cup \cdots \cup R_n) = \sum_{i=1}^n \mu(R_i)$ for disjoint $R_1, \ldots, R_n \in \mathcal{R}$.

Show that μ (which you may assume is well defined) is countably additive on $\mathcal{A}$, and deduce that there is a unique probability measure $\mathbb{P}$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ such that $\mathbb{P}[A_1 \times A_2] = \mathbb{P}_1[A_1]\mathbb{P}_2[A_2]$ for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.

(c) [12 marks] Define the *Borel σ-algebra* $\mathcal{B}$ on $[0, 1)$, and show that it is generated by $\mathcal{A} = \{[0, a) : 0 \leqslant a \leqslant 1\}$.

[*You may assume that if $U \subseteq [0, 1)$ is open as a subset of $[0, 1)$, then for each $x \in U$ there is a rational q and an $\epsilon > 0$ such that $x \in B_\epsilon(q) \subseteq U$, where $B_\epsilon(q) = \{y \in [0, 1) : |y-q| < \epsilon\}$.*]

For the rest of the question, you may assume that there is a probability measure μ on $([0, 1), \mathcal{B})$ such that $\mu([0, a)) = a$ for each $0 \leqslant a \leqslant 1$.

Given $A \subseteq [0, 1)$ and $0 < t < 1$, let $tA = \{tx : x \in A\}$. Show that for a fixed t, $\mathcal{C} = \{A \subseteq [0, 1) : tA \in \mathcal{B}\}$ is a σ-algebra. Deduce that if $A \in \mathcal{B}$ then $tA \in \mathcal{B}$, and show that $\mu(tA) = t\mu(A)$.

Let $A \in \mathcal{B}$. Define the *cone* C by $C = \{(x, y) \in [0, 1)^2 : x \in yA\}$. Show that $C \in \mathcal{B} \times \mathcal{B}$ and that $(\mu \times \mu)(C) = \mu(A)/2$.

[*Throughout this question standard properties of measure and integration may be assumed if stated correctly. In part (c) you may assume Fubini's Theorem.*]

2015 2. (a) [7 marks] Define the terms *filtration*, *martingale* and *stopping time*. Show carefully that if (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and τ is a stopping time, then $(M_{n\wedge\tau})$ is also a martingale.

(b) [13 marks] Let $(X_1, X_2, \ldots)$ be i.i.d. with $\mathbb{P}[X_n = 0] = \mathbb{P}[X_n = 1] = 1/2$. Let $\mathbf{a}$ be the sequence $(1, 0, 0, 1)$ and let $\mathbf{b} = (0, 1, 0, 1)$. Let $\tau_a = \inf\{n : (X_{n-3}, \ldots, X_n) = \mathbf{a}\}$, $\tau_b = \inf\{n : (X_{n-3}, \ldots, X_n) = \mathbf{b}\}$, and $\tau = \tau_a \wedge \tau_b$. Show carefully that τ_a, τ_b and τ are stopping times.

By showing that $\mathbb{P}[\tau > 4n] \leqslant (15/16)^n$ or otherwise, show that $\mathbb{E}[\tau] < \infty$.

Let A be the event $\{\tau = \tau_a\}$. By considering suitable martingales, find two linear relationships between $\mathbb{E}[\tau]$ and $\mathbb{P}[A]$, and hence find $\mathbb{E}[\tau]$ and $\mathbb{P}[A]$.

[*Descriptive definitions of the martingales will receive only partial credit; to obtain full marks, they must be defined precisely in terms of the variables X_n rather than in words. You may use any standard form of the Optional Stopping Theorem provided you state it correctly.*]

(c) [5 marks] Can the method of part (b) be applied with $\mathbf{a}$ and $\mathbf{b}$ replaced by any two distinct sequences of the same length r? For $r = 4$, what is the maximum possible value of $\mathbb{P}[A]$? Justify your answers briefly.

2015

3. (a) [12 marks] Carefully define what it means for $(M_{-n})_{n\geqslant 1}$ to be a *backwards martingale* with respect to a sequence $(\mathcal{F}_{-n})_{n\geqslant 1}$ of σ-algebras, including the relevant condition on $(\mathcal{F}_{-n})$.

What does it mean to say that a sequence $(X_n)_{n\geqslant 1}$ of random variables is *uniformly integrable*?

Is a backwards martingale necessarily uniformly integrable? What about a martingale? In each case give a proof or a counterexample.

[*You may assume that if X is integrable, then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbb{P}[A] < \delta$ implies $\mathbb{E}[|X|\mathbf{1}_A] < \epsilon$.*]

(b) [3 marks] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra generated by a π-system $\mathcal{I}$ with $\Omega \in \mathcal{I}$. Show that if X and Y are integrable random variables such that $\int_A X \, \mathrm{d}\mathbb{P} = \int_A Y \, \mathrm{d}\mathbb{P}$ for all $A \in \mathcal{I}$ and Y is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = Y$ a.s.

[*Standard properties of conditional expectation, and uniqueness results for measures, may be used without proof.*]

(c) [10 marks] Let $(X_n)_{n\geqslant 1}$ be a sequence of $\{0,1\}$-valued random variables with the following property: given $r \geqslant 0$ and $s \geqslant 0$, for distinct $i_1, \ldots, i_r, j_1, \ldots, j_s$ the probability that $X_{i_1} = \cdots = X_{i_r} = 1$ and $X_{j_1} = \cdots = X_{j_s} = 0$ is independent of the choice of $i_1, \ldots, i_r, j_1, \ldots, j_s$, and depends only on r and s.

Let $S_n = X_1 + \cdots + X_n$. Show, carefully justifying any symmetry argument, that $M_{-n} = S_n/n$ defines a backwards martingale with respect to $\mathcal{F}_{-n} = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$.

[*With n fixed, it may help to consider the family $\mathcal{I}$ of all events of the form $\{S_n = k, X_{n+1} = v_1, \ldots, X_{n+i} = v_i\}$, where $i \geqslant 0$, $0 \leqslant k \leqslant n$, and each $v_j \in \{0,1\}$.*]

Stating, without proof, any standard results concerning backwards martingales, deduce that S_n/n converges almost surely to a (random) limit P. What is $\mathbb{E}[X_1 \mid \sigma(P)]$? Justify your answer briefly.

2014 **1.** (a) [12 marks] Define the terms *π-system* and *σ-algebra*.

Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the σ-algebra *generated by* (X_n). What does it mean to say that the random variables $X_1, X_2, \ldots$ are *independent*?

State and prove Kolmogorov's 0/1-law.

[*You may assume that if two π-systems are independent, then so are the σ-algebras that they generate.*]

(b) [13 marks] Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent random variables, and let $S_n = X_1 + X_2 + \cdots + X_n$. State, with full justification, which of the following are tail events or tail random variables in terms of the sequence (X_n):

(i) $\limsup_{n\to\infty} S_n/n$,

(ii) $\limsup_{n\to\infty} S_n$,

(iii) $\{\omega \in \Omega : \sum_{n=1}^{\infty} X_n(\omega) < \infty\}$.

Let $X_1, X_2, \ldots$ be independent, non-negative random variables, and let $\mu_n = \mathbb{E}[X_n]$. Show that if $\sum_{n=1}^{\infty} \mu_n < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges almost surely to an almost surely finite random variable Y, and find $\mathbb{E}[Y]$. Now suppose that $\sum_{n=1}^{\infty} \mu_n = \infty$, and let E be the event $\{\sum_{n=1}^{\infty} X_n = \infty\}$. Give an example in which $\mathbb{P}[E] = 0$, and one in which $\mathbb{P}[E] = 1$.

[*In part (b) you may assume standard results from measure theory and integration theory without proof, provided they are correctly stated.*]

2014 **2.** (a) [15 marks] Define the terms *measurable space* and *finite measure*.

Let $\mathbb{P}$ and $\mathbb{Q}$ be finite measures on the measurable space $(\Omega, \mathcal{F})$. What does it mean to say that $\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$?

Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. What does it mean to say that Z is (a version of) the *conditional expectation* $\mathbb{E}[X \mid \mathcal{G}]$?

Show that the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ exists, and is unique up to equality almost surely.

[*You may assume the Radon–Nikodym Theorem without proof, provided it is correctly stated.*]

Show that if X is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = X$ a.s. Show that if $X_n \geq 0$ are integrable random variables with $X = \sum_{n=1}^{\infty} X_n$ integrable, then $\mathbb{E}[X \mid \mathcal{G}] = \sum_{n=1}^{\infty} \mathbb{E}[X_n \mid \mathcal{G}]$ a.s.

(b) [10 marks] Consider the probability space $([0,1), \mathcal{B}, \mathbb{P})$ where $\mathcal{B}$ is the Borel σ-algebra on $[0,1)$ and $\mathbb{P}$ is Lebesgue measure, and let X be the random variable defined by $X(\omega) = \omega$. For $n \geq 1$, let A_n be the event

$$\bigcup_{j=0}^{2^{n-1}-1} \left[\frac{j}{2^{n-1}} + \frac{1}{2^n}, \frac{j+1}{2^{n-1}}\right)$$

that the nth digit in the binary expansion of $\omega \in \Omega$ is 1, and let $\mathcal{F}_n = \sigma(\{A_n\}) = \{\emptyset, A_n, A_n^{\mathrm{c}}, \Omega\}$.

Show that A_n is independent of $\mathcal{F}_m$ for $m \neq n$. Find $\mathbb{E}[X \mid \mathcal{F}_n]$ and $\mathbb{E}[X \mid \mathcal{F}]$, where $\mathcal{F} = \sigma(\{A_1, A_3, A_5, A_7, \ldots\})$.

[*In part (b), standard properties of conditional expectation may be assumed without proof, provided they are stated correctly.*]

2014

3. (a) [8 marks] What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is *convex*?

Prove *Jensen's inequality* in the following form: if $f : \mathbb{R} \to [0, \infty)$ is convex and X is an integrable random variable, then $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$.

(b) [17 marks] Prove the following variant of Doob's Maximal Inequality: if (M_n) is a martingale and $\theta > 0$ then, for any (constant) $N \in \mathbb{N}$ and $\lambda > 0$,

$$\mathbb{P}\big[\max_{0 \leq n \leq N} M_n \geq \lambda\big] \leq \mathbb{E}[e^{\theta M_N}]e^{-\theta\lambda}.$$

[*The conditional form of Jensen's inequality and standard results about discrete stochastic integrals may be assumed without proof, if stated correctly.*]

Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = 1/2$ for all n, and set $S_n = X_1 + \ldots + X_n$. Show that for any $N \in \mathbb{N}$ and $t \geq 0$ we have

$$\mathbb{P}\big[\max_{0 \leq n \leq N} |S_n| \geq t\big] \leq 2e^{-t^2/(2N)}.$$

[*Hint: use series expansions to show that, for any* $\theta \in \mathbb{R}$, $(e^\theta + e^{-\theta})/2 \leq e^{\theta^2/2}$. *Standard results from measure theory may be assumed without proof.*]

2013

1. (a) Define the terms *σ-algebra*, *measure*, *probability space* and (real-valued) *random variable.*

(b) Let Ω be a set. What is a *π-system* on Ω? For $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a collection of subsets of Ω, define the *σ-algebra generated by* $\mathcal{A}$, $\sigma(\mathcal{A})$.

(c) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. What does it mean to say that σ-algebras $\mathcal{G}_n \subseteq \mathcal{F}$, $n = 1, 2, \ldots$, are *independent*? What does it mean to say that random variables X_n, $n = 1, 2, \ldots$, are *independent*?

Let $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ be π-systems on Ω with $\Omega \in \mathcal{I}_i \subseteq \mathcal{F}$ for each i, and let $\mathcal{G}_i = \sigma(\mathcal{I}_i)$. Show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent if and only if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$

for all $A \in \mathcal{I}_1$ and $B \in \mathcal{I}_2$.

Give, *with proof*, a corresponding condition for $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ to be independent, and deduce that random variables X, Y and Z on $(\Omega, \mathcal{F}, \mathbb{P})$ are independent if and only if

$$\mathbb{P}[X < x, Y < y \text{ and } Z < z] = \mathbb{P}[X < x]\,\mathbb{P}[Y < y]\,\mathbb{P}[Z < z]$$

for all $x, y, z \in \mathbb{R}$.

[*You may assume that, if $\mathcal{I}$ is a π-system on Ω and μ_1, μ_2 are measures such that $\mu_1(\Omega) = \mu_2(\Omega) < \infty$ and μ_1 and μ_2 agree on $\mathcal{I}$, then μ_1 and μ_2 agree on $\sigma(\mathcal{I})$. You may also assume that the Borel σ-algebra is generated by $\{\,(-\infty, x) : x \in \mathbb{R}\,\}$.*]

(d) Let X, Y and Z be real-valued random variables. Show that if X, Y and Z are independent, then $X + Y$ and Z are independent.

2013

2. (a) Let X and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra.

What does it mean to say that Y is *a version of the conditional expectation* $\mathbb{E}[X \mid \mathcal{G}]$? Show that if Y and Z are both versions of $\mathbb{E}[X \mid \mathcal{G}]$ then $Y = Z$ almost surely.

(b) Define the terms *filtration*, *martingale* and *stopping time.*

Prove the Optional Stopping Theorem in the following form: if $(M_n)_{n \geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 0}$ such that, for some real number L, for every $n \geqslant 1$ it is the case that $|M_n - M_{n-1}| \leqslant L$ almost surely, and τ is a stopping time with respect to $(\mathcal{F}_n)_{n \geqslant 0}$ with $\mathbb{E}[\tau] < \infty$, then $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

[*You may assume without proof that $(M_{n \wedge \tau})_{n \geqslant 0}$ is a martingale.*]

Briefly describe, without proof, an example showing that the condition $\mathbb{E}[\tau] < \infty$ cannot be omitted, even if $\mathbb{P}[\tau < \infty] = 1$.

(c) Let k be a positive integer. Let $(X_n)_{n \geqslant 1}$ be independent with

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = 1/2 \quad \text{for every } n,$$

let $S_n = \sum_{1 \leqslant i \leqslant n} X_i$, and let $\tau = \inf\{n : |S_n| = k\}$. By considering $M_n = S_n^2 - n$, or otherwise, show that $\mathbb{E}[\tau] = k^2$.

[*You may assume that $\mathbb{E}[\tau] < \infty$.*]

[*Throughout this question, standard properties of integration and of conditional expectation may be assumed if stated correctly.*]

2013

3. (a) Let $p \geqslant 1$, let $(X_n)_{n\geqslant 0}$ be a sequence of random variables, and let Y be a random variable. What does it mean to say (i) that $(X_n)_{n\geqslant 0}$ is *bounded in* L^p and (ii) that X_n *converges to* Y *in* L^p?

Show that if $1 \leqslant r < p$ and $(X_n)_{n\geqslant 0}$ is bounded in L^p, then it is bounded in L^r, and prove a similar statement relating convergence in L^p and in L^r.

[*You may assume Jensen's inequality without proof.*]

(b) Prove the *Pythagoras formula* that if $(M_n)_{n\geqslant 0}$ is a martingale in which $\mathbb{E}[M_n^2] < \infty$ for each n then for $-1 \leqslant i < j$

$$\mathbb{E}[(M_j - M_i)^2] = \sum_{k=i+1}^{j} \mathbb{E}[(M_k - M_{k-1})^2],$$

where M_{-1} is defined to be 0.

Deduce carefully that a martingale $(M_n)_{n\geqslant 0}$ is bounded in L^2 if and only if $\mathbb{E}[M_0^2] < \infty$ and $\sum_{k\geqslant 1} \mathbb{E}[(M_k - M_{k-1})^2] < \infty$.

[*Standard properties of conditional expectation and of* L^2 *spaces may be assumed if stated correctly.*]

(c) Assuming *without proof* Doob's Forward Convergence Theorem in the form that a martingale bounded in L^1 converges almost surely to an a.s. finite limit Y, show that a martingale $(M_n)_{n\geqslant 0}$ bounded in L^2 converges both almost surely and in L^2 to some random variable Y.

(d) Let $(X_n)_{n\geqslant 1}$ be independent with $\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = 1/2$ for each n. Let $M_n = \sum_{k=1}^{n} X_k/k$.

(i) Show that $(M_n)_{n\geqslant 0}$ is a martingale with respect to a suitable filtration.

(ii) Show that M_n converges almost surely to a random variable Y.

(iii) Find, *with proof*, $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$.

[*Throughout this question, standard properties of integration may be assumed without proof. You may assume that* $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.]

2012 **1.** (a) (i) Define the terms σ-*algebra*, *probability space*, and *event*.

(ii) Let $(A_n)_{n\geqslant 1}$ be a sequence of events in a probability space. What does it mean to say that A_n holds *infinitely often*?

(iii) State and prove the first and second Borel–Cantelli lemmas.

(b) Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking non-negative integer values. Suppose that for each n and each $i \geqslant 1$, $\mathbb{P}(X_n \geqslant i) = 1/i$.

(i) Evaluate

$$\mathbb{P}(\ X_n \geqslant n^\alpha \text{ infinitely often }\ )$$

for each (constant) real number $\alpha > 0$.

(ii) Show that the random variable

$$\limsup_{n\to\infty} \frac{\log X_n}{\log n}$$

is almost surely constant, and find its (almost sure) value.

(iii) Let $M_n = \max\{X_k : 1 \leqslant k \leqslant n\}$. Show that

$$\lim_{n\to\infty} \frac{\log M_n}{\log n} = 1$$

almost surely.

[*Throughout this question, basic properties of probability measures may be assumed without proof if stated correctly.*]

2012 **2.** (a) Let X, X_1, X_2 and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra.

(i) What does it mean to say that Y is $\mathcal{G}$-*measurable*? Define the *conditional expectation* $\mathbb{E}[X \mid \mathcal{G}]$, the σ-algebra *generated* by Y, and the conditional expectation $\mathbb{E}[X \mid Y]$.

(ii) Show that $\mathbb{E}[X_1 + X_2 \mid \mathcal{G}] = \mathbb{E}[X_1 \mid \mathcal{G}] + \mathbb{E}[X_2 \mid \mathcal{G}]$ almost surely.

(iii) What does it mean to say that X and Y are *independent*? Show that if X and Y are independent, then $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ almost surely. [*You may assume that if X and Z are independent random variables, then $\mathbb{E}[XZ] = \mathbb{E}[X]\mathbb{E}[Z]$.*]

(iv) Show that if Y is $\mathcal{G}$-measurable and XY is integrable, then $\mathbb{E}[XY \mid \mathcal{G}] = Y\mathbb{E}[X \mid \mathcal{G}]$. [*You may assume that if $Y \geqslant 0$ is a measurable function on Ω, then there is a sequence (Y_n) of simple functions with $Y_n \uparrow Y$.*]

(b) Let X, Y and Z be independent, with each uniformly distributed on the set $\{1, 2, \ldots, 6\}$, so that X, Y and Z represent numbers rolled on three dice. Evaluate

(i) $\mathbb{E}[X + YZ \mid Y]$,

(ii) $\mathbb{E}[X + Y \mid Y + Z]$, and

(iii) $\mathbb{E}[XY \mid Y + Z]$.

[*Standard properties of integration may be assumed throughout. In part* (b), *standard properties of conditional expectation may be assumed if stated correctly.*]

2012

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) What does it mean to say that

(i) $(\mathcal{F}_n)_{n\geqslant 0}$ is a *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$,

(ii) $(X_n)_{n\geqslant 0}$ is a *martingale* with respect to $(\mathcal{F}_n)_{n\geqslant 0}$?

(b) Doob's Upcrossing Lemma states that if $\mathbf{X} = (X_n)_{n\geqslant 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geqslant 0}$ and $a < b$ are real numbers, then for any $n \geqslant 0$,

$$\mathbb{E}[U_n([a,b], \mathbf{X})] \leqslant \frac{\mathbb{E}[(X_n - a)^-]}{(b-a)}.$$

(i) Define the number of *upcrossings* U_n.

(ii) Using Doob's Upcrossing Lemma (which you need not prove), deduce Doob's Forward Convergence Theorem in the following form: if $(X_n)_{n\geqslant 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geqslant 0}$ with the property that $\sup_n \mathbb{E}[|X_n|] < \infty$, then $(X_n)_{n\geqslant 0}$ converges almost surely to an almost surely finite limit X_∞.

(c) Let ε_n, $n \geqslant 1$, and V_n, $n \geqslant 0$, be independent random variables, with

$$\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2, \qquad \mathbb{P}(V_n = 1) = p_n, \qquad \mathbb{P}(V_n = 0) = 1 - p_n,$$

for all n. Define X_n inductively by $X_0 = 1$ and, for $n \geqslant 0$,

$$X_{n+1} = X_n + V_n \varepsilon_{n+1} \quad \text{if} \quad X_n > 0$$

and

$$X_{n+1} = 0 \quad \text{if} \quad X_n = 0.$$

(i) Show that $(X_n)_{n\geqslant 0}$ is a martingale with respect to a filtration that you should define.

(ii) Suppose that $p_n = 1$ for all n. Show that $X_n \to 0$ almost surely.

(iii) Now let $p_n = 1/(n+1)$ for all n. Does $X_n \to 0$ almost surely? What if $p_n = 1/(n+1)^2$ for all n? [*You may assume that for real numbers* $0 < x_n < 1$, $\prod_{n=1}^{\infty}(1 - x_n) = 0$ *if and only if* $\sum_{n=1}^{\infty} x_n = \infty$.]

[*Throughout this question, standard properties of integration and of conditional expectation may be used without proof if stated correctly.*]