B8.1: Probability, Measure and Martingales 2019
Problem Sheet 1

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Questions in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in. Questions are not in order of increasing difficulty: if you can't do one, try the next!*

Section 1 (Compulsory; mostly or entirely material from week 1. Some parts of some questions are revision!)

1. Let Ω be a set.
   (a) Suppose that $\{\mathcal{F}_j\}_{j\in J}$ is a non-empty family of σ-algebras on Ω. Prove that the intersection $\bigcap_{j\in J}\mathcal{F}_j$ is a σ-algebra on Ω.
   (b) Let $\mathcal{A}$ be a family of subsets of Ω. Show that there is a (clearly unique) minimal σ-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$; here minimality is with respect to inclusion: if $\mathcal{F}$ is a σ-algebra with $\mathcal{A}\subseteq\mathcal{F}$, then $\sigma(\mathcal{A})\subseteq\mathcal{F}$.
   (c) Give an example of two algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega=\{1,2,3\}$ whose union $\mathcal{F}_1\cup\mathcal{F}_2$ is *not* an algebra.

2. Let $\mathcal{P}=\{P_j\}_{j\in J}$ be a partition of a set Ω (i.e., a collection of disjoint non-empty sets with union Ω). Show that the set $\mathcal{U}(\mathcal{P})$ consisting of all possible unions of sets P_j is a σ-algebra on Ω. Conversely, show that any σ-algebra $\mathcal{F}$ on a *countable* set Ω is of the form $\mathcal{U}(\mathcal{P})$ for some partition $\mathcal{P}$ of Ω. [Warning: the converse is very far from true when Ω is uncountable.]

3. Let $\mathcal{T}$ be the set of all open subsets of $\mathbb{R}$, and let $\pi(\mathbb{R})$ be the π-system
$$\pi(\mathbb{R}) := \{(-\infty,x] : x\in\mathbb{R}\}\,.$$
   Prove that $\mathcal{B}(\mathbb{R})$ (defined to be $\sigma(\mathcal{T})$) is equal to $\sigma(\pi(\mathbb{R}))$. [Hint: recall that every open subset of $\mathbb{R}$ is a countable disjoint union of (finite or infinite) open intervals.]

4. Let $\mu:\mathcal{A}\to[0,\infty)$ be an additive set function on an algebra $\mathcal{A}$ taking only finite values. Show that μ is countably additive if and only if $\mu(A_n)\to 0$ for every sequence of sets in $\mathcal{A}$ with $A_n\downarrow\emptyset$.

5. Let Ω be a set and $\mathcal{I}$ a π-system on Ω. Let $\mathcal{F}=\sigma(\mathcal{I})$. Suppose that μ_1 and μ_2 are two measures on $(\Omega,\mathcal{F})$ with $\mu_1(\Omega)=\mu_2(\Omega)<\infty$ and $\mu_1=\mu_2$ on $\mathcal{I}$. Then Theorem 1.12 on uniqueness of extension says that $\mu_1=\mu_2$ on $\mathcal{F}$. Find an example on $\Omega=\{1,2,3,4\}$ where this fails if we drop the assumption that $\mathcal{I}$ is a π-system.

6. Revise the definition of Lebesgue outer measure $m^*(A)$ and the σ-algebra $\mathcal{M}_{\mathrm{Leb}}$ of Lebesgue-measurable sets.[1] Show that $A\subseteq[0,1]$ is Lebesgue measurable if and only if there exist a Borel

[1] The Lebesgue *outer measure* $m^*(A)$ of any set $A\subseteq\mathbb{R}$ is $\inf\{\sum_{i=1}^{\infty}|J_i| : J_1,J_2,\ldots$ are intervals with $\bigcup_{i=1}^{\infty}J_i\supseteq A\}$, and A is *Lebesgue measurable* if $m^*(F\cap A)+m^*(F\cap A^{\mathrm{c}})=m^*(F)$ for any $F\subseteq\mathbb{R}$.

set B and a null set N such that $A = B \,\Delta\, N$, where Δ denotes symmetric difference. Extend this to any $A \subseteq \mathbb{R}$.

7. Complete the proof of Example 1.7, showing that if Ω is countable then there is a bijection between measures on $(\Omega, \mathcal{P}(\Omega))$ and mass functions on Ω.

Section 2 (Extra practice questions, not for hand-in; some may need material from week 2)

A. Show that the following sets of subsets of $\mathbb{R}$ all generate the same σ-algebra:

$$\{(a,b) : a < b\}\,,\ \ \{(a,b] : a < b\}\,,\ \text{and}\ \ \{(-\infty, b] : b \in \mathbb{R}\}\,.$$

B. Given $\alpha < \beta$, let $\mathcal{A}$ be the algebra of finite unions of disjoint intervals of the form

$$A = (a_1, b_1] \cup \cdots \cup (a_n, b_n],\ \ \alpha \leqslant a_i < b_i \leqslant \beta,\ \ 1 \leqslant i \leqslant n,$$

and define, for every $A \in \mathcal{A}$, $\mu(A) := \sum_{i=1}^{n} (b_i - a_i)$.

Check that μ is well defined.

Prove that μ is countably additive on $\mathcal{A}$. [A related argument appears in lectures.]

C. Prove that, given $\alpha < \beta \in \mathbb{R}$, there is a unique Borel measure μ on $(\alpha, \beta]$ such that, for all $a, b \in (\alpha, \beta]$ with $a < b$,

$$\mu((a,b]) = b - a.$$

Prove also that there exists a unique such measure on the whole of $\mathbb{R}$.

D. Let $\mathcal{A}$ be a collection of subsets of a non-empty set Ω. Show that

(i) $\Omega \in \mathcal{A}$,

(ii) $A, B \in \mathcal{A},\ A \subseteq B \implies B \setminus A \in \mathcal{A}$, and

(iii) $A_n \in \mathcal{A},\ A_n \uparrow A \implies A \in \mathcal{A}$

hold (jointly, of course) if and only if

(a) $\Omega \in \mathcal{A}$,

(b) $A \in \mathcal{A} \implies A^{\mathrm{c}} \in \mathcal{A}$, and

(c) $\mathcal{A}$ is closed under countable disjoint unions.

Either condition defines what is variously called a *d-system*, *λ-system*, or *Dynkin system*.

Show that if μ_1, μ_2 are measures on $(\Omega, \mathcal{F})$ with $\mu_1(\Omega) = \mu_2(\Omega) < \infty$, then $\{A : \mu_1(A) = \mu_2(A)\}$ is a Dynkin system.

Show that if $\mathcal{A}$ is a Dynkin system and a π-system, then it is a σ-algebra.

**[Very hard] Show that the Dynkin system generated by a π-system $\mathcal{A}$ (i.e., the intersection of all Dynkin systems containing $\mathcal{A}$) is a π-system and so a σ-algebra. This is what is needed to prove Theorem 1.12.