B8.1: Probability, Measure and Martingales 2019
Problem Sheet 2

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in. Questions are not in order of increasing difficulty: if you can't do one, try the next!*

Section 1 (Compulsory)

1. Let (A_n) be a sequence of independent events. Show that if I, J are disjoint (finite or countable) sets, then
$$\mathbb{P}\left[\bigcap_{i\in I} A_i \cap \bigcap_{i\in J} A_i^{\mathrm{c}}\right] = \prod_{i\in I} \mathbb{P}[A_i] \prod_{i\in J}(1-\mathbb{P}[A_i]).$$
(You may wish to use induction on $|J|$.)

2. Prove Scheffé's Lemma: let (f_n) be a sequence of non-negative integrable functions on $(\Omega, \mathcal{F}, \mu)$ and suppose that $f_n(x) \to f(x)$ for μ-almost every $x \in \Omega$ where f is also integrable. Then
$$\int |f_n - f| \,\mathrm{d}\mu \to 0 \quad \textit{iff} \quad \int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu.$$

3. Let $F : \mathbb{R} \to [0,1]$ be an increasing, right-continuous function with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

   The goal of this exercise is to provide an alternative proof to that outlined in lectures that there exists a unique probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that, for all x,
$$\mu((-\infty, x]) = F(x). \tag{1}$$

   (a) Show that, if μ exists, it is unique.

   (b) Define $G : (0,1) \to \mathbb{R}$ by
$$G(y) := \inf\{x \in \mathbb{R} : F(x) \geqslant y\}.$$
   Show that G is increasing, that
$$F(x) \geqslant y \iff x \geqslant G(y),$$
   and deduce that G is left-continuous (i.e., $G(y) \to G(x)$ as $y \uparrow x$).

   (c) Let $\Omega := (0,1)$, $\mathcal{F} := \mathcal{B}((0,1))$ be the Borel σ-algebra on $(0,1)$, and let $\mathbb{P}$ be the restriction of Lebesgue measure to $\mathcal{F}$. Show that $X := G$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that the induced measure $\mu_X := \mathbb{P}\circ X^{-1}$ is such that
$$\mu_X((-\infty, x]) = \mathbb{P}[X \leqslant x] = F(x).$$

4. Let $X_1, X_2, \ldots$ be independent uniformly distributed random variables on $[0,1]$. Let A_n be the event that a record occurs at time n:

$$A_n = \{ \ X_n > X_m \text{ for all } m < n \ \}.$$

Find the probability of A_n and show that $A_1, A_2, \ldots$ are independent. Deduce that, with probability one, infinitely many records occur.

Now consider *double records*, that is two records in a row. What is the probability of infinitely many double records?

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed real-valued random variables such that $\mathbb{P}[X_n = 1] = p$, $\mathbb{P}[X_n = -1] = 1 - p$ where $p \neq 1/2$. Let $S_0 := 0$ and, for all $n \geqslant 1$,

$$S_n := \sum_{k=1}^{n} X_k.$$

Show that the probability of the event $\{S_n = 0 \text{ i.o.}\}$ is zero.

[You might, or might not, find Stirling's formula useful: $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$.]

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed real-valued random variables such that

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = 1/2.$$

Let $S_0 := 0$ and, for all $n \geqslant 1$, $S_n := \sum_{k=1}^{n} X_k$.

For $x \in \mathbb{Z}$ let

$$A_x := \{S_n = x \text{ for infinitely many } n\},$$
$$B_- := \big\{\liminf_{n\to\infty} S_n = -\infty\big\}, \quad B_+ := \big\{\limsup_{n\to\infty} S_n = \infty\big\}.$$

(a) Using Kolmogorov's 0-1 law, prove that $\mathbb{P}[B_-] \in \{0,1\}$ and $\mathbb{P}[B_+] \in \{0,1\}$.

(b) Prove that $\mathbb{P}[B_-] = \mathbb{P}[B_+]$.

(c) Using a Borel–Cantelli lemma show that, for all $k \geqslant 1$,

$$\limsup_{n\to\infty}(S_{n+k} - S_n) = k \ \text{ a.s.}$$

(d) Deduce from (c) that $\mathbb{P}[B_-^{\mathrm{c}} \cap B_+^{\mathrm{c}}] = 0$, and therefore that $\mathbb{P}[B_-] = \mathbb{P}[B_+] = 1$. Conclude that, for all $x \in \mathbb{Z}$, $\mathbb{P}[A_x] = 1$.

7. Let Z be a random variable taking non-negative integer values. Show that $\mathbb{E}[Z] = \sum_{n=1}^{\infty} \mathbb{P}[Z \geqslant n]$.

Let X be an integrable random variable (that is $\mathbb{E}[|X|] < \infty$). Show that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}[|X| \geqslant \varepsilon n] < \infty.$$

Now let $(X_i)_{i\geqslant 1}$ be a sequence of identically distributed random variables and let $M_n = \max\{|X_j| : 1 \leqslant j \leqslant n\}$. If $\mathbb{E}[|X_1|^\alpha] < \infty$ for some $\alpha \in (0, \infty)$, then show that as $n \to \infty$,

$$\frac{M_n}{n^{1/\alpha}} \to 0 \quad \text{with probability 1.}$$

[Hint: Fix $\varepsilon > 0$. Let $A_n = \{|X_n| \geqslant \varepsilon n^{1/\alpha}\}$ and apply the first Borel–Cantelli Lemma.]

Section 2 (Extra practice questions, not for hand-in)

A. Convince yourself that the proofs of the convergence theorems for Part A Integration carry over to the general measure spaces that we have introduced.

B. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces, and $(\Omega, \mathcal{F})$ their product. Check carefully that the set $\mathcal{A}$ of finite disjoint unions of measurable rectangles is indeed an algebra on Ω. [Thinking about atoms may help arrive at a clean answer]

C. Let (X_n) be a sequence of independent identically distributed random variables taking values in $\{R, G\}$ (R for red and G for green), with

$$\mathbb{P}[X_n = R] = 1 - \mathbb{P}[X_n = G] = \alpha$$

for some $\alpha \in (0, 1)$.

Show that, almost surely, each configuration has no one-colour arithmetic subsequence, i.e., that for all $a, b \in \mathbb{N}$, $a \neq 0$, we have $\{X_{an+b} : n \in \mathbb{N}\} = \{R, G\}$.

Now find a deterministic sequence (x_n) satisfying the same property.

D. Let X_1, X_2 be independent exponentially distributed random variables with parameter one. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\} - Y_1$. Show that Y_1 and Y_2 are independent. Generalize this to the case of three independent exponential random variables with parameter one.

E. Let Ω denote the probability space $([0, 1], \mathcal{B}([0, 1]), \mathbb{P})$, where $\mathbb{P}$ is Lebesgue measure. Show that there is a sequence (X_n) of independent random variables on Ω with $\mathbb{P}[X_n = 0] = \mathbb{P}[X_n = 1] = 1/2$ for every n. Show also that given any such sequence (X_n) on any probability space, the random variable $Y = \sum_{n=1}^{\infty} X_n/2^n$ has the uniform distribution on $[0, 1]$.

Deduce that there exists a sequence (Y_n) of independent random variables on Ω each with the uniform distribution on $[0, 1]$. Hence show that there is a product probability measure $\mathbb{P}$ on the countable product of $([0, 1], \mathcal{B}([0, 1]))$ with itself such that $\mathbb{P}[\omega_i \leqslant x] = x$ for all $i \geqslant 1$ and all $x \in [0, 1]$, where ω_i denotes the ith coordinate of a point $\omega = (\omega_i)_{i \geqslant 1}$ of the product space.