B8.1: Probability, Measure and Martingales 2019
Problem Sheet 3

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in. Questions are not in order of increasing difficulty: if you can't do one, try the next!*

Section 1 (Compulsory)

1. Let $(X_n)_{n\geqslant 2}$ be a sequence of independent random variables such that

$$\mathbb{P}[X_n = n] = \mathbb{P}[X_n = -n] = \frac{1}{2n\log n}; \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{n\log n}.$$

   Let $S_n = X_2 + \ldots + X_n$. Prove that $\frac{S_n}{n} \to 0$ in probability, but not almost surely.

   [Hint: calculate the variance of S_n to show the convergence in probability, and use a Borel–Cantelli lemma to consider the almost sure convergence].

2. (a) Roll a fair die until we get a six. Let Y be the total number of rolls and X the number of 1's. Show that

$$\mathbb{E}[X \mid Y] = \frac{1}{5}(Y-1) \quad \text{and} \quad \mathbb{E}[X^2 \mid Y] = \frac{1}{25}(Y^2 + 2Y - 3).$$

   (b) Consider two independent Poisson processes $N^{(1)}(t), t \geqslant 0$ and $N^{(2)}(t), t \geqslant 0$.
   Let $T = \inf\{t : N^{(1)}_t > 0\}$ be the time of the first point of the first process.
   Let $X = N^{(2)}(T^{(1)}_1)$ be the number of points of the second process which occur before the first point of the first process.
   What are $\mathbb{E}(X|T)$ and $\mathbb{E}(X^2|T)$?

3. Let X and Y be bounded random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose joint density is some measurable function $f(x,y)$. (That is, for measurable $A \subseteq \mathbb{R}^2$, $\mathbb{P}[(X,Y) \in A] = \int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.) Apply Fubini's Theorem (Theorem 2.21 in the notes) to show that

$$\mathbb{E}[X \mid \sigma(Y)] = \frac{\int x f(x,Y)\,\mathrm{d}x}{\int f(x,Y)\,\mathrm{d}x}.$$

   In other words, $\mathbb{E}[X \mid \sigma(Y)] = g(Y)$, where $g : \mathbb{R} \to \mathbb{R}$ is the function

$$g(y) = \frac{\int x f(x,y)\,\mathrm{d}x}{\int f(x,y)\,\mathrm{d}x}.$$

4. Let X and Y be bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that each of the following statements implies the next, but that the reverse implications do not hold: (i) X and Y are independent; (ii) $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ a.s.; (iii) $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

5. Let X be a $\sigma(Y)$-measurable random variable. Show that there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $X = f(Y)$. [Hint: first consider the case when $X = \mathbf{1}_A$ for some suitable set A.] What can you say when X is $\sigma(Y, Z)$-measurable?

6. (a) Show that if Y is a $\mathcal{G}$-measurable random variable, then $\mathbb{E}[Y(X - \mathbb{E}(X|\mathcal{G}))] = 0$.
   (b) Suppose $\mathcal{H} \subset \mathcal{G}$ are σ-algebras and X is a random variable with $\mathbb{E}(X^2) < \infty$. Show that

$$\mathbb{E}\left[\{X - \mathbb{E}(X|\mathcal{G})\}^2\right] + \mathbb{E}\left[\{\mathbb{E}(X|\mathcal{G}) - \mathbb{E}(X|\mathcal{H})\}^2\right] = \mathbb{E}\left[\{X - \mathbb{E}(X|\mathcal{H})\}^2\right].$$

7. Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables with $\mathbb{E}[X_n] = 1$ for all n. Define

$$M_0 = 1, \qquad M_n = \prod_{i=1}^{n} X_i \quad \text{for } n \geqslant 1.$$

   Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that $(M_n)_{n\geqslant 0}$ is a martingale w.r.t. $(\mathcal{F}_n)$.

   Let $(Y_i)_{i\geqslant 1}$ be a sequence of independent, identically distributed random variables and set $S_n = \sum_{i=1}^{n} Y_i$. Show that for any t such that the moment generating function $\psi(t) = \mathbb{E}[e^{tY_1}]$ is finite,

$$M_n = \frac{e^{tS_n}}{(\psi(t))^n}, \qquad n \geqslant 0$$

   defines a martingale. It is frequently called the *exponential martingale.*

8. Use the Monotone Convergence Theorem (MCT) to prove the conditional form of the MCT, and the conditional form of the MCT to prove the conditional form of Fatou's Lemma. (See Proposition 4.8 in the notes.)

Section 2 (Extra practice questions, not for hand-in)

A. Let $X_1, X_2, \ldots$ be independent with $\mathbb{E}X_i = 0$ for each i, and $\mathrm{Var}(X_i) = \sigma_i^2 < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$ and let $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$. Show that $S_n^2 - s_n^2$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

B. Show that the sequence (X_n) of random variables defined in question 1 converges to (the deterministic value) $X = 0$ in probability and in L^1, but not in L^p for any $p > 1$.

C. (Hölder's Inequality) Suppose that p and q are non-negative real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X and Y be real-valued random variables with $\mathbb{E}[|X|^p] < \infty$ and $\mathbb{E}[|Y|^q] < \infty$. Show that $\mathbb{E}[|XY|] \leqslant (\mathbb{E}[|X|^p])^{1/p} (\mathbb{E}[|Y|^q])^{1/q}$.

D. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a random variable with $\mathbb{E}[X^2] = 1$ and $\mathbb{E}[X^4] < \infty$. Show that $\mathbb{E}[|X|] \geqslant 1/\sqrt{\mathbb{E}[X^4]}$. [Hint: write $X^2 = |X|^r |X|^{2-r}$, choose r conveniently and exploit Hölder's inequality.]

E. Let $X \geqslant 0$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Show that $\{X > 0\} \subseteq \{\mathbb{E}[X \mid \mathcal{G}] > 0\}$, up to an event of zero probability. Show also that $\{\mathbb{E}[X \mid \mathcal{G}] > 0\}$ is the smallest $\mathcal{G}$-measurable event that contains the event $\{X > 0\}$, up to zero probability events.