B8.1: Probability, Measure and Martingales 2019
Problem Sheet 4

Lecturer: James Martin

The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in. Questions are not in order of increasing difficulty: if you can't do one, try the next!*

Section 1 (To be handed in for the class)

1. Let S and T be stopping times with respect to $(\mathcal{F}_n)_{n\geqslant 0}$. Prove that $S+T$, $\max(S,T)$ and $\min(S,T)$ are stopping times too. Assuming it is always non-negative, must $S-T$ be a stopping time?

2. Suppose that τ is a stopping time such that for some $K \geqslant 1$ and some $\varepsilon > 0$, we have, for every $n \geqslant 0$:
$$\mathbb{P}[\tau \leqslant n+K \mid \mathcal{F}_n] \geqslant \varepsilon \text{ a.s.}$$
Prove by induction that, for all $m \in \mathbb{N}$,
$$\mathbb{P}[\tau > mK] \leqslant (1-\varepsilon)^m.$$
Deduce that $\mathbb{E}[\tau] < \infty$.

3. Suppose $X_n, n \geqslant 1$ are i.i.d. with finite mean. Suppose N is a stopping time for the sequence X_n, with $\mathbb{E}(N) < \infty$. Show that
$$\mathbb{E}\left(\sum_{n=1}^{N} X_n\right) = \mathbb{E}(N)\mathbb{E}(X_1).$$
(This is known as *Wald's equation*.)

4. (Martingale formulation of Bellman's Optimality Principle)

Your winnings per unit stake on a certain game are ε_n in round n, where $(\varepsilon_n)_{n\geqslant 1}$ is an i.i.d. sequence of random variables with
$$\mathbb{P}[\varepsilon_n = +1] = p, \quad \mathbb{P}[\varepsilon_n = -1] = q, \quad \text{where } 1/2 < p = 1-q < 1.$$
Let $\mathcal{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$. Assume that your stake V_n on game n is $\mathcal{F}_{n-1}$-measurable (i.e. only depends on the outcome of the game up to time $n-1$), and that V_n must lie strictly between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your object is to maximize the expected "interest rate" at a certain integer time horizon N, i.e. $\mathbb{E}[\log(Z_N/Z_0)]$, given the constant $Z_0 > 0$, your fortune at time 0.

Prove that, if $Z_n > 0$,
$$\mathbb{E}[\log(Z_{n+1}/Z_n) \mid \mathcal{F}_n] = f(V_{n+1}/Z_n),$$

where $f(x) := p\log(1+x) + q\log(1-x)$.

Deduce that, if (V_n) is any (predictable) strategy, then $(\log Z_n - n\alpha)_{n\geqslant 0}$ is a supermartingale, where

$$\alpha := p\log p + q\log q + \log 2;$$

α is sometimes called the "entropy". Conclude that

$$\mathbb{E}[\log(Z_N/Z_0)] \leqslant N\alpha.$$

Describe explicitly the strategy (V_n) for which $(\log Z_n - n\alpha)_{n\geqslant 0}$ becomes a *martingale* and the inequality becomes an equality.

5. Let (Z_n) be a Galton–Watson branching process with offspring distribution X, with $\mu = \mathbb{E}[X] > 1$ and $\sigma^2 = \mathrm{Var}[X] < \infty$, and set $M_n = Z_n/\mu^n$. Using induction, find a formula for $\mathbb{E}[Z_n^2]$ in terms of n, μ and σ. Deduce that (M_n) is bounded in $\mathcal{L}^2$, and converges in $\mathcal{L}^2$ and in $\mathcal{L}^1$.

6. Let $A \subset \mathbb{Z}^2$ be a finite set of points in the square lattice, and let B (the *boundary* of A) be the set of points in $\mathbb{Z}^2 \setminus A$ with at least one (horizontal or vertical) neighbour in A. Show that given any function $g : B \to \mathbb{R}$ there is a function $f : (A \cup B) \to \mathbb{R}$ such that $f|_B = g$ and, for every $v \in A$,

$$f(v) = \frac{1}{4}\sum_{w\sim v} f(w),$$

where the sum is over the 4 neighbours of w; f is called a *discrete harmonic function* with boundary condition g.

[*Hint.* For $v \in A$ consider a random walk starting at v, and consider the value of g at the point where the walk first hits the boundary.]

7. Let $\mathcal{C}$ be a family of random variables. Show that

$$\mathcal{C} \text{ is bounded in } \mathcal{L}^p \text{ for some } p > 1 \implies \mathcal{C} \text{ is uniformly integrable} \implies \mathcal{C} \text{ is bounded in } \mathcal{L}^1.$$

Show that the reverse implications don't hold.

8. *Gambler's Ruin.* Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence of random variables with

$$\mathbb{P}[X_1 = 1] = p, \quad \mathbb{P}[X_1 = -1] = q, \quad \text{where } 0 < p = 1 - q < 1,$$

and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + \ldots + X_n, \quad T := \inf\{n : S_n = 0 \text{ or } S_n = b\}.$$

Let $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. Explain why T satisfies the conditions in Question 2. Prove that

$$M_n := \left(\frac{q}{p}\right)^{S_n} \text{ and } N_n := S_n - n(p-q)$$

are martingales w.r.t. $(\mathcal{F}_n)$. Deduce the values of $\mathbb{P}[S_T = 0]$ and $\mathbb{E}[T]$. (In Prelims, we found these by solving linear recurrence relations.)

9. Consider a sequence of independent tosses of a coin. Let p be the probability of a head on any toss. Let A be the hypothesis that $p = a$, and let B be the hypothesis that $p = b$, where $a, b \in (0,1)$. Let X_i denote the outcome of the ith toss, and let

$$Z_n = \frac{P_A(X_1, X_2, \ldots, X_n)}{P_B(X_1, X_2, \ldots, X_n)},$$

where P_A and P_B denote probabilities of sequences under the hypotheses A and B respectively.

Show that if B is true, then Z_n is a martingale, and $Z_\infty := \lim Z_n$ exists with probability 1.

If $b \neq a$, what is the distribution of Z_∞?

This is a special case of the *consistency of the likelihood ratio test* in Statistics.

Section 2 (Extra questions for practice/revision/extension – not for hand-in. Solutions will be provided at the end of term.)

A. For a Galton–Watson branching process with offspring generating function $f(\theta) = \mathbb{E}[\theta^X]$, show that $W_n = q^{Z_n}$ defines a martingale, where q is the smallest root of $\theta = f(\theta)$ in $[0,1]$.

B. Let $M_n, n \geqslant 0$ be a martingale with $M_0 = 0$. Which of the following are possible?

(a) For some n, $\mathbb{E}(M_n) > 0$.

(b) For some a.s. finite stopping time τ, $\mathbb{E}(M_\tau) > 0$.

(c) $M_n \to \infty$ as $n \to \infty$ with probability 1.

What about if in addition M is bounded in $\mathcal{L}^1$? What about if in addition M is uniformly integrable?

C. Suppose that we repeatedly flip a fair coin. What is the expected number of throws until we see the pattern HTHTHT for the first time? What are the maximum and minimum expected waiting times for patterns of length 6? Give an example of a pattern attaining the maximum and one attaining the minimum.

D. Let $(X_n, n \geqslant 1)$ be a sequence of independent random variables, with $\mathbb{E}(X_n) = 0$ for every n, and $\sum_{n=1}^\infty \mathbb{E}(X_n^2) < \infty$. Show that $\sum_{n=1}^\infty X_n$ converges with probability 1.

E. Show that the probability that the fraction of white balls in Polya's Urn is ever as large as $3/4$ is at most $2/3$. Is it in fact equal to $2/3$, or smaller?

F. Show that if $\mathcal{F}_n$ is a filtration with $\mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_n)$, and $Y_n \to Y \in \mathcal{L}^1$, then $\mathbb{E}(Y_n \mid \mathcal{F}_n) \to \mathbb{E}(Y \mid \mathcal{F}_\infty)$ in $\mathcal{L}^1$.

G. Let $(Y_n, n \geqslant 1)$ be a sequence of independent, identically distributed random variables. Let $t \in \mathbb{R}$ and suppose that the value $\psi(t) = \mathbb{E}(e^{tY_1})$ of the moment generating function is finite. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n Y_k$. Let

$$M_n = \frac{e^{tS_n}}{\psi(t)^n}.$$

Show that M_n is a martingale (it is called an *exponential martingale*). Is it necessarily the case that M_n converges to a limit as $n \to \infty$?

H. Consider a Markov chain $X_n, n \geqslant 0$ on a finite state space S. Let g be a function from S to $\mathbb{R}$, and let $M_n = g(X_n)$.

(a) Suppose the chain is irreducible (i.e. S consists of a single communicating class). Show that M is a martingale only in the trivial case where g is a constant function.

(b) In general, what functions g give martingales? (Consider how g must behave on each closed communicating class.)

(c) Show that (a) may fail in the case of a countably infinite state space.

(This question is closely related to Q6.)