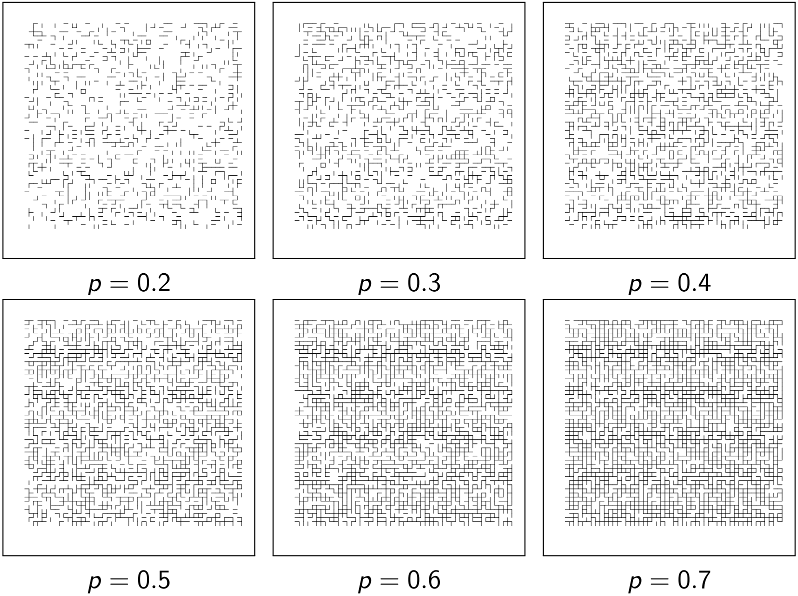
Percolation

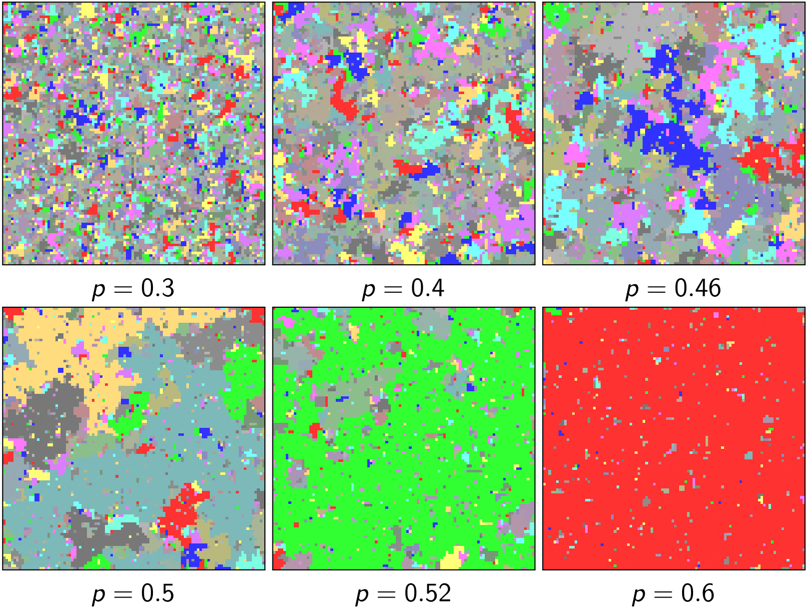
Two-dimensional lattice $\mathbb{Z}^2$. Let $p \in (0, 1)$.

Let each edge of the lattice be open with probability p and closed with probability $1 - p$, independently for different edges.

What does the connected component $C(v)$ of a given vertex $v \in \mathbb{Z}^2$ look like? (i.e. the set of vertices reachable from v by following open edges)

Does there exist an infinite connected component?


$p = 0.2$
$p = 0.3$
$p = 0.4$
$p = 0.5$
$p = 0.6$
$p = 0.7$


$p = 0.3$
$p = 0.4$
$p = 0.46$
$p = 0.5$
$p = 0.52$
$p = 0.6$

The events that

$$\{\text{there exists an infinite cluster}\}$$
$$\{\text{there exist infinitely many infinite clusters}\}$$

are tail events. Changing the state of any finite collection of edges does not change whether these events occur. So by the Kolmogorov 0-1 law, these events must have probability 0 or 1.

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Using methods we'll see later in this lecture:

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There exists p_c with $0 < p_c < 1$ such that:

(i) *If $p < p_c$ then with probability 1 there is no infinite cluster.*

(ii) *If $p > p_c$ then with probability 1 there is infinite cluster.*

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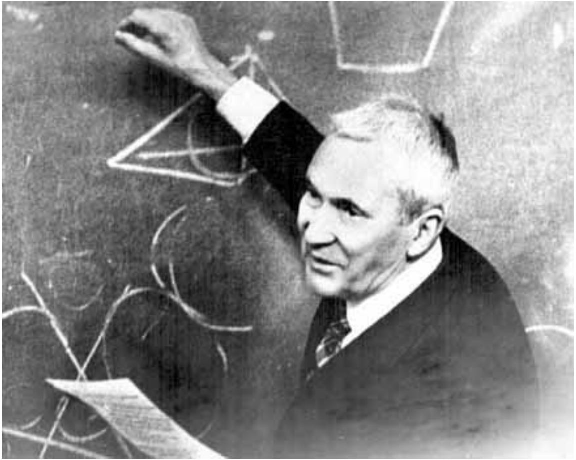
Somewhat harder to show: in fact $p_c = 1/2$.

Theorem

(i) *If $p \leq 1/2$ then with probability 1 there is no infinite cluster.*

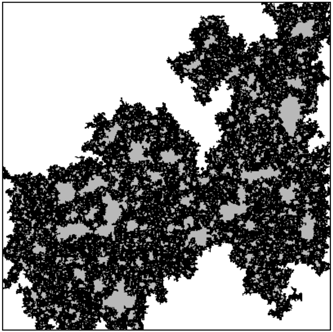
(ii) *If $p > 1/2$ then with probability 1 there is a unique infinite cluster.*



Andrei Kolmogorov (1903-1987)

Critical behaviour: $p = 1/2$

Fractal picture ("scale-invariant"). Conformal invariance. Interfaces described by "Schramm-Loewner evolution" and related processes. "Universal" behaviour – not dependent on particular lattice structure (but only known rigorously in certain very specific cases!)



Schramm, Lawler, Werner (Fields 2006), Smirnov (Fields 2010).

Critical exponents: size of largest cluster in $\Lambda_L \sim L^{91/48}$;
$\mathbb{P}(|C(v)| = n) \sim n^{-96/91}$; $\mathbb{P}(v \leftrightarrow w) \sim |v - w|^{-5/24}$...

Next year....:

Part C Probability on Graphs and Lattices (Stats Dept. course)

Part C Introduction to Schramm-Loewner Evolution (Maths Dept. course)