B8.1: Probability, Measure and Martingales 2019
Problem Sheet 4
Solutions to optional questions

A. Since $q \in [0,1]$ and $Z_n \geqslant 0$ we have that $q^{Z_n} \in [0,1]$, so integrability is clear.

$$\mathbb{E}\left[q^{Z_{n+1}} \,\middle|\, \mathcal{F}_n\right] = \mathbb{E}\left[q^{X_1^{(n+1)}+\cdots+X_{Z_n}^{(n+1)}} \,\middle|\, \mathcal{F}_n\right] = \mathbb{E}[f(q)^{Z_n} \mid \mathcal{F}_n] = f(q)^{Z_n} = q^{Z_n}.$$

B. (a) Certainly not. One of the first things we proved about martingales (part (2) of Proposition 5.4) is that $\mathbb{E}(M_n) = \mathbb{E}(M_0)$ for all n.

(b) Simple example: simple symmetric random walk on $\mathbb{Z}$ started from 0, with τ the first hitting time of 1.

Boundedness in $\mathcal{L}^1$ is not enough for the conclusion of the Optional Stopping Theorem to apply. Indeed, we have seen several examples of martingales Z with $Z_0 > 0$ which are non-negative (hence are bounded in $\mathcal{L}^1$) and such that Z hits 0 in finite time almost surely. For example, let Z be a branching process with mean offspring size equal to 1 or let $Z_n = Y_1 Y_2 \dots Y_n$ where Y_i takes value 0 or 2 with probability 1/2 each (the "double or quits" martingale). Then putting $M_n = 1 - Z_n$ and τ the first hitting time of 0 by the process Z gives an example.

However, uniform integrability *is* enough for the conclusion of the Optional Stopping Theorem to apply, as discussed in Lecture 15. So in that case $\mathbb{E}(M_\tau) = 0$ for all stopping times τ (we may even allow $\tau = \infty$).

(c) Yes, $M_n \to \infty$ a.s. is possible. For example, let $M_n = \sum_{k=1}^n X_k$ where X_k are independent and X_k takes value 1 with probability $(k^2-1)/k^2$, and value $-(k^2-1)$ with probability $1/k^2$.

Then each X_k has mean 0 so M is a martingale. But by applying Borel-Cantelli, with probability 1, $\mathbb{P}(X_k = 1 \text{ for all large enough } k) = 1$. So indeed $M_n \to \infty$ with probability 1.

If M is bounded in $\mathcal{L}^1$, we know from the Forward Convergence Theorem that M converges a.s. to a finite limit. So $M_n \to \infty$ is not possible.

C. We mimic Example 6.6 in the notes (or equivalently the ABRACADABRA version in the lectures). We assume a fair casino in which each player starts with a fortune of £1. Player i bets £1 that the outcome of the ith flip is H. If she wins then she bets her £2 that the outcome of the $(i+1)$st flip is T. If she wins again, then she bets her £4 that the outcome of the $(i+2)$nd flip is H. And so on. The casino's profits form a martingale in which the changes are bounded by a constant (note that at most 6 people bet at one time).

We stop after the τth flip (when we have first seen the pattern HTHTHT for the first time). By Lemma 6.5 of lectures (see Question 2 on this sheet), $\mathbb{E}[\tau] < \infty$ and so the expected casino profit at time τ is zero (Doob's Optional Stopping Theorem). At this time player $\tau - 5$ has a fortune of £64, player $\tau - 3$ has a fortune of £16 and player $\tau - 1$ has a fortune of £4. The change in fortune of all players at time τ is then £$(64 + 16 + 4 - \tau)$ which has mean zero giving $\mathbb{E}[\tau] = 84$.

More formally, for $k \geqslant 1$ and $n \geqslant 0$ define

$$M_n^{(k)} = \begin{cases} 0 & n < k \\ 2^{n-k+1}\mathbf{1}_{\{X_k=1, X_{k+1}=0, \ldots, X_n=a\}} - 1 & k \leqslant n \leqslant k+5 \\ 2^6\mathbf{1}_{\{X_k=1, X_{k+1}=0, \ldots, X_n=0\}} - 1 & n \geqslant k+6 \end{cases}$$

where the X_i are independent Bernoulli random variables with success probability $1/2$. For each k the sequence $M_n^{(k)}$ is a martingale in n (the winnings of the kth gambler after n coins have been revealed), and $M_n = \sum_{k \geqslant 1} M_n^{(k)}$ is the required martingale. (Note that an infinite sum of martingales need not be a martingale, or indeed make sense, but here only finitely many k participate for a given n.)

The maximum expected waiting time is $64+32+16+8+4+2 = 126$ for HHHHHH or TTTTTT. The minimum is 64, for example for HHHHHT.

D. There are more elementary methods but let's write a solution using what we know about martingales.

Let $M_n = \sum_{i=1}^n X_i$. Then M_n is a martingale, and $\mathbb{E}(M_n^2) = \mathrm{Var}(M_n) = \sum_{i=1}^n \mathrm{Var}(X_i)$ (by independence of the X_i), which is bounded.

So M is bounded in $\mathcal{L}^2$. This is much stronger than we need for a.s. convergence; for example boundedness in $\mathcal{L}^1$ is enough (by the Forward Convergence Theorem). So indeed M converges almost surely. Since M is bounded in $\mathcal{L}^2$ we could also deduce that M also converges in $\mathcal{L}^2$.

E. Let W_n be the proportion of white balls in Polya's Urn. Let S_n be the same process stopped the first time that it reaches $3/4$ or greater (if that ever happens).

The process S starts at $1/2$, and is a bounded martingale, and so converges a.s. and in $\mathcal{L}^1$ to some limit S_∞ with $\mathbb{E}S_\infty = 1/2$.

If the proportion of white balls ever reaches $3/4$ or more, then $S_\infty \geqslant 3/4$.

Since S_∞ is non-negative, Markov's inequality gives $\mathbb{P}(S_\infty \geqslant 3/4) \leqslant \frac{1/2}{3/4} = 2/3$.

(This bound is not sharp. For Markov's inequality to be tight, we would need that S_∞ only takes the values 0 and $3/4$. But certainly the limit can be between 0 and $3/4$, and also the limit can be greater than $3/4$, since the process W can jump over $3/4$ to some higher value.)

F. To show

$$\mathbb{E}(Y_n \mid \mathcal{F}_n) \to \mathbb{E}(Y \mid \mathcal{F}_\infty) \text{ in } \mathcal{L}^1,$$

it's enough that

$$\mathbb{E}(Y_n \mid \mathcal{F}_n) - \mathbb{E}(Y \mid \mathcal{F}_n) \to 0 \text{ in } \mathcal{L}^1,$$

and

$$\mathbb{E}(Y \mid \mathcal{F}_n) \to \mathbb{E}(Y \mid \mathcal{F}_\infty) \text{ in } \mathcal{L}^1.$$

For the first,

$$\begin{aligned} \mathbb{E}\left[|\mathbb{E}(Y_n \mid \mathcal{F}_n) - \mathbb{E}(Y \mid \mathcal{F}_n)|\right] &\leqslant \mathbb{E}\left[E(|Y_n - Y| \mid \mathcal{F}_n\right] \\ &= \mathbb{E}|Y_n - Y| \\ &\to 0. \end{aligned}$$

For the second: this is the setting of Theorem 8.7; $\mathbb{E}(Y \mid \mathcal{F}_n)$ is a Doob martingale, whose limit a.s. and in $\mathcal{L}^1$ is $\mathbb{E}(Y \mid \mathcal{F}_\infty)$.

G. $M_n = \prod_{i=1}^n X_i$ where $X_i = e^{tY_i}/\psi(t)$.

The variables X_i are i.i.d. with mean 1, so indeed M_n is a martingale.

It's a non-negative martingale so it converges a.s. The limit M_∞ is a.s. equal to 0 (for M to converge to a strictly positive limit, it would require X_i to converge to 1, which is a.s. not the case).

H. (a) Many ways to approach this. If a finite Markov chain has a single communicating class, then the class is recurrent, and each state $x \in S$ is reached infinitely often with probability 1. The process $g(X_n)$ is bounded, so if it's a martingale, it converges a.s. But it can only do so if $g(x)$ is the same for all $x \in S$.

(b) Assume that every state may be reached with positive probability (otherwise we may ignore the states which can never be reached – the function g can take arbitrary values there).

By the same argument in (a), the function g must be constant on any closed communicating class.

Let the closed classes be $C_1, C_2, \ldots, C_k$ for some $k \geqslant 1$. Fix $g(x) = a_i$ for all $x \in C_i$.

We see that $g(X_n)$ converges a.s. to some X_∞: specifically, $X_\infty = a_i$ if X is absorbed in C_i. We have that $g(X_n)$ is bounded, hence uniformly integrable, so if it's a martingale, then $g(X_0) = \mathbb{E}[g(X_\infty)]$. We must have $g(x) = \sum_i a_i \mathbb{P}_x(\text{reach } C_i)$ (where $\mathbb{P}_x$ denotes the law of the Markov chain started from point x).

So there is a k-dimensional space of functions g that give martingales. Once the values $a_i, i = 1, \ldots, k$ are specified, i.e. the values of g at points in $C_1, \ldots, C_k$, the remaining values of g are forced.

(c) We could consider for example a nearest-neighbour random walk on $\mathbb{Z}$, with a drift away from the origin: say $p_{x,x+1} = 1 - p_{x,x-1} = 2/3$ for $x > 0$, and $p_{x,x+1} = 1 - p_{x,x-1} = 1/3$ for $x \leqslant 0$.

Started from any state, this walk tends to $-\infty$ with positive probability, and to $+\infty$ with positive probability.

We may set for example $g(x) = \mathbb{P}_x(X_n \to \infty)$. Then the process $g(X_n)$ is a martingale, but it is not constant. The "limits" $\pm\infty$ play the role of the absorbing classes C_i in part (b). So although the chain has a single communicating class, it has two different possible limiting behaviours, and non-constant martingales are possible. (The most general form of the function g is in fact now $g(x) = a\mathbb{P}_x(X_n \to \infty) + b\mathbb{P}_x(X_n \to -\infty)$.)