Part B Classical Mechanics: Problem Sheet 1 (of 4)

1. Consider a closed system consisting of N point particles with masses m_I, position vectors $\mathbf{r}_I$ in an inertial frame $\mathcal{S}$, such that particle J exerts a force $\mathbf{F}_{IJ}$ on particle I for $I \neq J$.

   (a) Explain why Newton's third law and Galilean invariance means there exists an inertial frame $\mathcal{S}_0$ where the centre of mass

$$\mathbf{R} = \frac{\sum_{I=1}^{N} m_I \mathbf{r}_I}{\sum_{I=1}^{N} m_I}$$

   is at rest at the origin.

   (b) Suppose now that there exist functions $V_{IJ} = V_{JI} = V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$, depending only on the distances $|\mathbf{r}_I - \mathbf{r}_J|$ between pairs of particles, such that $\mathbf{F}_{IJ} = -\partial_{\mathbf{r}_I} V_{IJ}$. Show that the total angular momentum and total energy

$$E = \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 + \sum_{I<J} V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$$

   are conserved.

2. Consider the one-dimensional harmonic oscillator with action $S[q(t)] = \int_0^{2\pi} L(q, \dot{q}) \, \mathrm{d}t$, Lagrangian

$$L = L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m q^2 ,$$

   and with boundary conditions $q(0) = q(2\pi) = 1$. Determine the Lagrange equation of motion, and show that a solution is $q(t) = \cos t$. Is this solution unique? By considering the two paths $q_1(t) = 1$, $q_2(t) = \cos 2t$ show that the critical function $q(t)$ is neither a maximum nor a minimum of the action S.

3. Consider the change of generalized coordinates $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$. Show that

$$\dot{q}_a = \sum_{b=1}^{n} \frac{\partial q_a}{\partial \tilde{q}_b} \dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t}.$$

   Defining

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \equiv L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) ,$$

   show that the Lagrange equations in the two coordinate systems are related via

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a} \right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} = \sum_{b=1}^{n} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_b} \right) - \frac{\partial L}{\partial q_b} \right] \frac{\partial q_b}{\partial \tilde{q}_a} .$$

   Hence conclude that the Lagrange equations take the same form in all coordinate systems.

4. Consider the purely kinetic Lagrangian

$$L = T = \frac{1}{2} \sum_{a,b=1}^{n} g_{ab}(\mathbf{q}) \dot{q}_a \dot{q}_b ,$$

where we assume that the symmetric matrix $g_{ab} = g_{ba}$ depends on the generalized coordinates $\mathbf{q}$ and is invertible at each point in configuration space. Show that Lagrange's equations take the form

$$\ddot{q}_a + \sum_{b,c=1}^{n} \Gamma^a_{bc} \dot{q}_b \dot{q}_c \;=\; 0 \;, \qquad a \;=\; 1, \ldots, n \;,$$

where

$$\Gamma^a_{bc} \;\equiv\; \frac{1}{2} \sum_{d=1}^{n} (g^{-1})^{ad} \left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d} \right) \;.$$

The matrix of functions $g_{ab}(\mathbf{q})$ defines a *metric* on configuration space, and the Lagrange equation is known as the *geodesic equation*.

5. Consider a system of two light rods of equal length, smoothly jointed together, with the other two ends of the rods fixed at two points in a horizontal plane. A mass m is attached to the point where the rods are jointed. Introduce an appropriate generalized coordinate for the system, and determine the Lagrangian.

6. A *double pendulum* consists of a simple pendulum of mass m_1 and length l_1 pivoted at the origin, together with another simple pendulum of mass m_2 and length l_2, pivoted at the mass m_1. The whole system moves freely in a vertical plane under gravity. If θ_1 and θ_2 denote the angles each pendulum makes with the vertical, show that the Lagrangian is

$$L \;=\; \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right]$$
$$+ m_1 g l_1 \cos \theta_1 + m_2 g (l_2 \cos \theta_2 + l_1 \cos \theta_1) \;.$$

7. (*Optional: this question is included for interest.*) Consider the central inverse square law force Lagrangian

$$L \;=\; \frac{1}{2} m |\dot{\mathbf{r}}|^2 + \frac{\kappa}{r} \;,$$

where $r = |\mathbf{r}|$ and κ is a constant.

(a) Show by direct computation that the vector

$$\mathbf{A} \;=\; \mathbf{p} \wedge \mathbf{L} - m \kappa \frac{\mathbf{r}}{r}$$

is conserved, where $\mathbf{p} = m \dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ are momentum and angular momentum about the origin, respectively. [*Hint*: You may find it helpful to write $\dot{\mathbf{p}} = h(r) \frac{\mathbf{r}}{r}$ and show $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \wedge \mathbf{L}) \;=\; -mh(r) r^2 \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathbf{r}}{r})$.]

(b) Show that $\mathbf{A} \cdot \mathbf{L} = 0$ and $|\mathbf{A}|^2 = 2mE|\mathbf{L}|^2 + m^2 \kappa^2$, where E is the conserved energy. Explain why $\mathbf{A}$ is a constant vector in the plane of the orbit.

(c) By taking the dot product $\mathbf{A} \cdot \mathbf{r}$ derive the orbit equation

$$\frac{1}{r} \;=\; \frac{m \kappa}{|\mathbf{L}|^2} \left(1 + \frac{|\mathbf{A}|}{m \kappa} \cos \theta \right) \;,$$

where θ denotes the angle between $\mathbf{r}$ and $\mathbf{A}$. Notice we have found the orbit without solving any differential equation! The eccentricity is $|\mathbf{A}|/m\kappa$.

Please send comments and corrections to lmason@maths.ox.ac.uk.