Part B Classical Mechanics: Problem Sheet 2 (of 4)

1. A smooth rod OP is forced to rotate with angular speed ω about a fixed vertical axis OQ so that it makes a constant angle α with OQ, where $0 < \alpha < \frac{\pi}{2}$. A ring of mass m slides freely on the rod OP.

   (a) Using a coordinate r for the distance of the ring to O, find the Lagrangian for the dynamics of the ring. Deduce that, as long as the ring remains on the rod, the quantity

   $$H = \frac{1}{2}m(\dot{r}^2 - \omega^2 r^2 \sin^2\alpha) + mgr\cos\alpha$$

   is a constant of the motion.

   (b) The ring is projected from O towards P with initial speed $\frac{\lambda g}{\omega}\cot\alpha$. Show that if $0 < \lambda < 1$ the ring can never exceed a maximum value of r. Setting $\lambda = 1$ find $r(t)$.

2. Consider N point particles with masses m_I, position vectors $\mathbf{r}_I$ and Lagrangian

   $$L = \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 - V(\{|\mathbf{r}_I - \mathbf{r}_J|\}) .$$

   In particular the potential function V depends only on the distances between the particles. Show that the Galilean boost $\mathbf{r}_I \to \mathbf{r}_I + \epsilon \mathbf{v} t$ (for all $I = 1, \ldots, N$) is a symmetry of L, and hence identify the function f that enters the form of Noether's theorem given in lectures. Does this lead to a new conserved quantity?

3. In lectures we considered the example of a bead of mass m sliding freely on a circular wire of radius a, with the wire forced to rotate about a vertical diameter with constant angular frequency $\dot{\varphi} = \omega$. The effective Lagrangian for the angle $\theta = \theta(t)$ is

   $$L_1(\theta, \dot{\theta}) = \frac{1}{2} m a^2 \dot{\theta}^2 - V_{\text{eff}}(\theta) ,$$

   where

   $$V_{\text{eff}}(\theta) = mga\cos\theta - \frac{1}{2} m a^2 \omega^2 \sin^2\theta .$$

   Show that $\theta = 0$, $\theta = \pi$ and (for $\omega \geq \sqrt{g/a}$) $\theta = \theta_0$ given by

   $$\theta_0 = \cos^{-1}\left(-\frac{g}{\omega^2 a}\right)$$

   are equilibria, and determine which are stable and which are unstable.

4. A light string is stretched to a tension τ between two fixed points A and B, a distance $3a$ apart, on a smooth horizontal table. Two point masses, each of mass m, are attached to the string at the points P_1, P_2. In equilibrium the four points A, P_1, P_2 and B are all equal distances apart. The system is set to perform small transversal oscillations by displacing transversely the two masses. Given that the potential energy is $\tau\Delta$, where Δ is the extension of the string from equilibrium and τ is constant, find the normal frequencies and normal modes of the vibration. Sketch the normal modes. [*Hint*: You might find it helpful to introduce generalized coordinates x, y such that $A = (0, 0)$, $P_1 = (a, x)$, $P_2 = (2a, y)$, $B = (3a, 0)$.]

5. A bead of mass m_1 slides without friction on a fixed horizontal wire which occupies the interval $[-a, a]$ of the x-axis. A light spring, of spring constant k, connects the bead to the point $-a$, and a second light spring, with the same spring constant, connects the bead to the point a. A massless rod of length l hangs freely from the bead and its other end carries a particle of mass m_2. The motion is restricted to the vertical plane containing the wire.

   (a) Show that the Lagrangian for the system is

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + 2l\cos\theta\,\dot{x}\dot{\theta} + l^2\dot{\theta}^2) - kx^2 + m_2 gl\cos\theta \ ,$$

   where x is the position of the bead on the wire, θ is the angle between the rod and the downward vertical, and g is the acceleration due to gravity.

   (b) Find a constant of the motion.

   (c) Determine the Lagrange equations of motion and show there are two equilibrium points.

   (d) Write down the quadratic Lagrangian for small oscillations about the point of stable equilibrium.

   (e) Find the normal frequencies when $g = l = k = m_1 = m_2 = 1$.

6. (a) Let $\mathcal{S}$, $\hat{\mathcal{S}}$ and $\mathcal{S}'$ be three reference frames, all with the same origin O. If the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is $\boldsymbol{\omega}$, and in turn the angular velocity of $\hat{\mathcal{S}}$ relative to $\mathcal{S}'$ is $\hat{\boldsymbol{\omega}}$, show that $\mathcal{S}$ has angular velocity $\boldsymbol{\omega} + \hat{\boldsymbol{\omega}}$ relative to $\mathcal{S}'$. [*Hint*: This is most easily shown using the Coriolis formula.]

   (b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a time-dependent orthonormal basis. Explain why $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_1 = 0$, and give two similar relations. Hence deduce that for some $\alpha, \beta, \gamma, \lambda, \mu, \nu$,

$$\dot{\mathbf{e}}_1 = \beta\mathbf{e}_3 - \gamma\mathbf{e}_2 \ , \qquad \dot{\mathbf{e}}_2 = \lambda\mathbf{e}_1 - \alpha\mathbf{e}_3 \ , \qquad \dot{\mathbf{e}}_3 = \mu\mathbf{e}_2 - \nu\mathbf{e}_1 \ .$$

   Next show that $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + \dot{\mathbf{e}}_2 \cdot \mathbf{e}_1 = 0$, and deduce from this and similar identities that $\lambda = \gamma$, $\mu = \alpha$, and $\beta = \nu$. Hence deduce that there exists a vector $\boldsymbol{\omega}$ such that $\dot{\mathbf{e}}_i = \boldsymbol{\omega} \wedge \mathbf{e}_i$ holds for all $i = 1, 2, 3$. [*This is an alternative way to introduce the angular velocity vector* $\boldsymbol{\omega}$.]

Please send comments and corrections to `lmason@maths.ox.ac.uk`.