Part B Classical Mechanics: Problem Sheet 3 (of 4)

1. Find the principal moments of inertia at the centre of mass of an ellipsoid of mass M and uniform density, bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 ,$$

where (x, y, z) are standard Cartesian coordinates. Using your result hence deduce the inertia tensor of a uniform sphere at a point on its surface.

2. (a) Show that none of the three principal moments of inertia can exceed the sum of the other two.

   (b) By introducing cylindrical polar coordinates, show that the centre of mass of an axi-symmetric body lies on the axis of symmetry. Show that the axis of symmetry is a principal axis, and that if we take this to be the $\mathbf{e}_3$ direction in an orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ then the other two principal moments of inertia satisfy $I_1 = I_2$.

   (c) Consider a rigid straight rod of line density ρ. Taking the rod to lie along the $\mathbf{e}_3$ direction, show that the inertia tensor at any point on the rod in this basis is diagonal, with eigenvalues $I_1 = I_2$, $I_3 = 0$. More generally when is it possible to have zero as a principal moment of inertia?

3. Consider the system of Euler equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 , \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 , \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 , \end{aligned}$$

describing the free rotation of a rigid body with principal moments of inertia I_1, I_2, I_3. Suppose that $I_1 < I_2$, $I_3 = I_1 + I_2$ and the body is set in motion with $\omega_2 = 0$ and $\omega_3\sqrt{I_2 + I_1} = \omega_1\sqrt{I_2 - I_1}$. Show that

$$\dot{\omega}_1^2 = \left(\frac{I_2 - I_1}{I_2 + I_1}\right)\omega_1^2\left(\frac{2T}{I_2} - \omega_1^2\right) ,$$

where T is the conserved kinetic energy. Hence find a solution of the form $\omega_1(t) = c_1 \operatorname{sech}(c_2 t)$ for appropriate constants c_1, c_2. What happens as $t \to \infty$?

4. Consider a rotating rigid body with centre of mass coordinates (x, y, z), principal moments of inertia I_1, I_2, I_3 about the centre of mass, and mass M. Explain why, in terms of Euler angles (θ, φ, ψ), the kinetic energy of the body is

$$\begin{aligned} T = {} & \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}I_1(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)^2 \\ & + \frac{1}{2}I_2(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 . \end{aligned}$$

[*This is bookwork, so this question is asking you to work through the derivation for yourself, to make sure you understand it!*]

5. (a) Show that the non-zero principal moment of inertia of a uniform rigid rod of mass M and length L about either end is $I = \frac{1}{3}ML^2$.
   (b) A *compound pendulum* is constructed by pivoting the rigid rod in part (a) about one end at the origin O. The rod swings freely in a vertical plane under gravity. If θ denotes the angle the rod makes with the vertical, show that the kinetic energy of the rod is

$$T = \frac{1}{6}ML^2\dot{\theta}^2 .$$

   Show that the frequency of small oscillations about the point of stable equilibrium is $\omega = \sqrt{3g/2L}$. How does this compare with a simple pendulum of the same mass and length?

6. A thin uniform disc of radius a and mass M moves on a smooth horizontal table, touching the table at a point of its circumference. Introduce Cartesian coordinates (x, y, z) for the centre of mass of the disc, and Euler angles (θ, φ, ψ) to describe its orientation, with the $\mathbf{e}_3$ axis parallel to the axis of symmetry of the disc, so that θ gives the angle between the plane of the disc and the table.
   (a) Explain why there are five degrees of freedom for this system, and why you can choose $(x, y, \theta, \varphi, \psi)$ as generalized coordinates.
   (b) Write down the Lagrangian. Show that $\dot{x}$ and $\dot{y}$ are both constant.
   (c) Initially the disc has spin n about its axis of symmetry, which makes an angle α with the vertical, while $\dot{\theta} = 0 = \dot{\varphi}$. Show that, in the subsequent motion, the spin around the axis is constant and

$$a\dot{\theta}^2(1 + 4\cos^2\theta) + 4an^2(\cos\alpha - \cos\theta)^2(\sin\theta)^{-2} + 8g(\sin\theta - \sin\alpha) = 0 .$$

7. Determine the Hamiltonian for the Lagrange top.

Please send comments and corrections to `lmason@maths.ox.ac.uk`.