Part B Classical Mechanics: Problem Sheet 4 (of 4)

1. Show that the Poisson bracket relations

$$\{\mathbf{q}, f\} = \frac{\partial f}{\partial \mathbf{p}} \qquad \text{and} \qquad \{\mathbf{p}, f\} = -\frac{\partial f}{\partial \mathbf{q}}$$

hold for any function f on phase space.

2. (a) Show that the angular momentum vector $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ satisfies the Poisson bracket relation $\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$ [*Hint*: you may use the identity $\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$]. Deduce that $\{\mathbf{L}, |\mathbf{L}|^2\} = \mathbf{0}$.

   (b) Show that for a particle moving in a central potential $V = V(|\mathbf{r}|)$ we have $\{\mathbf{L}, H\} = 0$. Deduce that angular momentum is conserved.

   (c) Recall that the Laplace-Runge-Lenz vector is defined as

$$\mathbf{A} \equiv \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} ,$$

   where κ is a constant. Derive the Poisson bracket relations $\{L_i, A_j\} = \sum_{k=1}^{3} \epsilon_{ijk} A_k$.

   (d) (*Optional: for the enthusiast.*) Assuming that the Hamiltonian is $H = |\mathbf{p}|^2/2m - \kappa/|\mathbf{r}|$, show that $\{\mathbf{A}, H\} = \mathbf{0}$ and deduce that $\mathbf{A}$ is conserved.

3. (a) Show that $Q = \arctan \frac{q}{p}$, $P = \frac{1}{2}(p^2 + q^2)$ is a canonical transformation.

   (b) For which functions $f(q)$ does $Q = f(q)\,\mathrm{e}^{t} \cos p$, $P = f(q)\,\mathrm{e}^{-t} \sin p$ define a canonical transformation?

4. Let

$$\mathcal{D}_f \equiv \sum_{\alpha=1}^{2n} \frac{\partial f}{\partial y_\alpha} \Omega_{\alpha\beta} \frac{\partial}{\partial y_\beta}$$

be the Hamiltonian vector field associated to the function $f = f(\mathbf{y})$ on phase space, with coordinates $\mathbf{y} = (y_1, \ldots, y_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and where Ω is the symplectic matrix introduced in lectures.

   (a) Show that for all functions f, g, h on phase space we have

$$\mathcal{D}_f(\mathcal{D}_g\, h) - \mathcal{D}_g(\mathcal{D}_f\, h) = \mathcal{D}_{\{f,g\}}\, h .$$

   (b) Hence prove the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 .$$

   (c) Consider the one-dimensional harmonic oscillator Hamiltonian $H(q, p) = \frac{1}{2}(p^2 + q^2)$. Show that

$$\mathrm{e}^{-t\mathcal{D}_H}\, q = q \cos t + p \sin t , \qquad \mathrm{e}^{-t\mathcal{D}_H}\, p = p \cos t - q \sin t .$$

   Deduce that $Q = q \cos s + p \sin s$, $P = p \cos s - q \sin s$ defines a canonical transformation for any s. What happens for $s = \pi/2$ and $s = \pi$?

5. Consider the time-dependent Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2, t) \ ,$$

where the potential $V(q_1, q_2, t) = U(\tilde{q}_1, \tilde{q}_2)$ and $\tilde{q}_1 = q_1 \cos t - q_2 \sin t$, $\tilde{q}_2 = q_1 \sin t + q_2 \cos t$. Determine the canonical transformation defined by the generating function of the second kind

$$F_2(q_1, q_2, P_1, P_2, t) = \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} .$$

In particular show that the transformed Hamiltonian K is independent of time t, and may be written as $K = \tilde{H} + \Lambda$ where $\Lambda = \Lambda(Q_1, Q_2, P_1, P_2, t)$ satisfies equation (5.70) in the lecture notes.

6. Consider the *elliptic cylindrical coordinates* on $\mathbb{R}^3$, related to Cartesian coordinates (x, y, z) by

$$x = a \cosh \mu \cos \nu \ , \qquad y = a \sinh \mu \sin \nu \ , \qquad z = z \ .$$

Show that the Hamiltonian for a particle of mass m moving in a potential V is given in these coordinates by

$$H = \frac{1}{2ma^2(\sinh^2 \mu + \sin^2 \nu)}(p_\mu^2 + p_\nu^2) + \frac{1}{2m} p_z^2 + V(\mu, \nu, z) \ .$$

Assuming the potential V takes the form

$$V(\mu, \nu, z) = \frac{V_\mu(\mu) + V_\nu(\nu)}{\sinh^2 \mu + \sin^2 \nu} + V_z(z) \ ,$$

show that the Hamilton-Jacobi equation is completely separable, and determine the three corresponding ordinary differential equations.

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