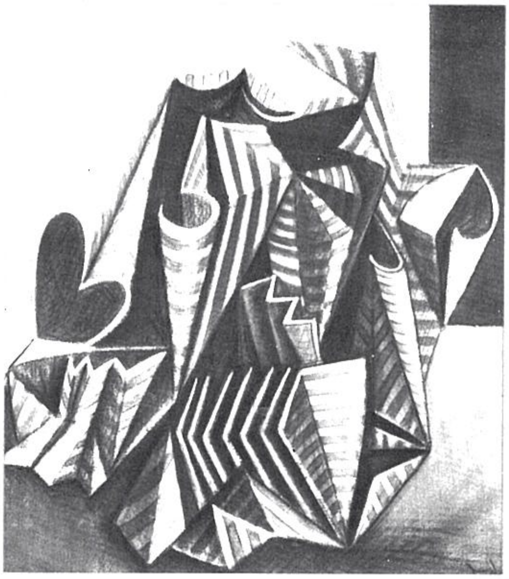
LINEAR PROGRAMMING



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A verification of this claim, left for problem 10.1, amounts to tedious pl
grinding.

Since Chapter 2, we have been referring to dictionaries (10.7) as *feasible*
for all $r \in B$. Now we shall refer to (10.7) as *dual-feasible* if the correspond
dictionary (10.8) is feasible. Thus, (10.7) is dual-feasible if and only if $\bar{c}_s \leq$
$s \in N$.

THE DUAL SIMPLEX METHOD

As observed in Chapter 5, every LP problem in the standard form may be solved
applying the simplex method to its dual; that is, an optimal solution of the pr
problem may be read off the optimal dictionary for the dual problem. The obser-
tions made in the last section imply that this strategy may be implemented with
any reference to the dual problem: the sequence of feasible dictionaries created
the simplex method working on the dual problem may be represented by a seque
of dual-feasible dictionaries associated with the primal problem. The resulti
algorithm, designed by C. E. Lemke (1954) and known as the *dual simplex meth*
constitutes a valuable tool of sensitivity analysis.

An explicit description of the dual simplex method follows mechanically from
fact that a dual variable y_k is basic in (10.8) if and only if the corresponding prim
variable x_k is nonbasic in (10.7). Thus, a dual dictionary arising from (10.8) by
single pivot, with y_i entering and y_j leaving the basis, will correspond to the prim
dictionary arising from (10.7) by a single pivot, with x_i leaving and x_j entering
basis. In particular, if (10.8) is one of the feasible dictionaries created by the simpl
method, then y_i and y_j are determined by the familiar rules: the choice of y_i is mo-
vated by the desire to increase $-w$ and the choice of y_j is dictated by the need
preserve feasibility when y_i increases. Formally, i may be any subscript $i \in B$ w

$$\bar{b}_i < 0 \qquad (10.9)$$

and j must be a subscript $j \in N$ that has

$$\bar{a}_{ij} < 0 \quad \text{and} \quad \bar{c}_j/\bar{a}_{ij} \leq \bar{c}_s/\bar{a}_{is} \qquad \text{for all} \quad s \in N \quad \text{with} \quad \bar{a}_{is} < 0. \qquad (10.10)$$

Hence an iteration of the dual simplex method, beginning with a dual-feasi
dictionary (10.7), consists of first choosing a subscript $i \in B$ that satisfies (10.9), the
finding a subscript $j \in N$ that satisfies (10.10), and finally pivoting, with x_i leavi
and x_j entering the basis.

For illustration, suppose that (10.7) reads

$$\begin{aligned}
x_1 &= -4 + 3x_2 - 11x_4 + x_5\\
x_3 &= 3 - x_2 + 3x_4 - 2x_5\\
\hline
z &= 12 - 4x_2 - x_4 - x_5.
\end{aligned}$$

To choose the leaving variable, we examine $\bar{b}_1 = -4$ and $\bar{b}_2 = 3$; the leaving variable must be x_1. To find the entering variable, we examine the ratios $\bar{c}_2/\bar{a}_{12} = 4/3$ and $\bar{c}_5/\bar{a}_{15} = 1$ (ignoring $\bar{c}_4/\bar{a}_{14}$ since $\bar{a}_{14} \geq 0$); the entering variable must be x_5. The pivot with x_5 entering and x_1 leaving the basis yields the dual-feasible dictionary

$$\begin{array}{rcl} x_5 &=& -5 - 2x_1 + 5x_2 - 19x_4 \\ x_3 &=& 4 + x_1 - 3x_2 + 11x_4 \\ \hline z &=& 8 - x_1 - x_2 - 12x_4. \end{array}$$

In general, each iteration of the dual simplex method is nothing but a disguised version of an iteration of the simplex method working on the dual problem. This particular example disguises the iteration that leads from

$$\begin{array}{rcl} y_2 &=& 4 - 3y_1 + y_3 \\ y_4 &=& 1 + 11y_1 - 3y_3 \\ y_5 &=& 1 - y_1 + 2y_3 \\ \hline -w &=& -12 + 4y_1 - 3y_3 \end{array} \quad \text{to} \quad \begin{array}{rcl} y_1 &=& 1 + 2y_3 - y_5 \\ y_2 &=& 1 - 5y_3 + 3y_5 \\ y_4 &=& 12 + 19y_3 - 11y_5 \\ \hline -w &=& -8 + 5y_3 - 4y_5. \end{array}$$

If no $i \in B$ satisfies (10.9), then the computations terminate: since dictionary (10.7) is not only dual-feasible but also feasible, it describes an optimal solution $x_1^*, x_2^*, \ldots,$ defined by $x_s^* = 0$ for all $s \in N$ and $x_r^* = \bar{b}_r$ for all $r \in B$. Similarly, if no $j \in N$ satisfies (10.10), then the computations terminate: since the dual problem is unbounded, the primal problem is infeasible. The latter conclusion may be also reached directly, without any reference to the dual problem: if no $j \in N$ satisfies (10.10), then $\bar{a}_{is} \geq 0$ for all $s \in N$, and so the right-hand side of the equation

$$x_i = \bar{b}_i - \sum_{s \in N} \bar{a}_{is} x_s,$$

with $\bar{b}_i < 0$, assumes a negative value whenever the values of all $x_s (s \in N)$ are nonnegative.

In Chapter 7, we saw how the simplex method may be implemented in the revised format without the use of dictionaries. The dual simplex method may be implemented similarly, with only the numbers $\bar{b}_r (r \in B)$ and $\bar{c}_s (s \in N)$ stored and updated in each iteration. Working out the details amounts to answering the following four questions:

(i) How is the leaving variable x_i found?

(ii) How is the entering variable x_j found?

(iii) How are the numbers $\bar{b}_r$ updated?

(iv) How are the numbers $\bar{c}_s$ updated?

The first question does not present any difficulty since the numbers $\bar{b}_r (r \in B)$ featured in (10.9) are readily available.

To answer the second question, we need only find a way of recovering the numbers $\bar{a}_{is} (s \in N)$ featured in (10.10) from the original data. For this purpose, let us recall that the first m rows of dictionary (10.7) may be recorded (in the notation of Chapter 7) as

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{A}_N\mathbf{x}_N. \tag{10.11}$$

If the leaving variable x_i appears in the pth position of the basis heading, then it is the pth equation in (10.11) that expresses x_i in terms of the nonbasic variables, and so it is the pth row of the matrix $\mathbf{B}^{-1}\mathbf{A}_N$ that consists of the desired numbers $\bar{a}_{is}$. But the pth row of $\mathbf{B}^{-1}\mathbf{A}_N$ equals $\mathbf{v}\mathbf{A}_N$ with $\mathbf{v}$ standing for the pth row of $\mathbf{B}^{-1}$; in turn, $\mathbf{v}$ itself may be found by solving the system $\mathbf{vB} = \mathbf{e}$ with $\mathbf{e}$ standing for the pth row of the $m \times m$ identity matrix. Hence the numbers $\bar{a}_{is}(s \in N)$ may be computed by first solving the system $\mathbf{vB} = \mathbf{e}$ and then computing the row vector $\mathbf{w}_N = \mathbf{v}\mathbf{A}_N$; each $\bar{a}_{is}$ is a component w_s of this vector $\mathbf{w}_N$.

To answer the third question, we need only recall that the numbers $\bar{b}_r(r \in B)$ are nothing but the components $x_r^*(r \in B)$ of the vector $\mathbf{x}^* = [x_1^*, x_2^*, \ldots, x_{n+m}^*]^T$ associated with dictionary (10.7); in fact, (10.11) may be written as

$$\mathbf{x}_B = \mathbf{x}_B^* - \mathbf{B}^{-1}\mathbf{A}_N\mathbf{x}_N.$$

These numbers may be updated as in Chapter 7: having determined the entering variable x_j, and therefore the entering column $\mathbf{a}$, we solve the system $\mathbf{Bd} = \mathbf{a}$ and replace $\mathbf{x}_B^*$ by $\mathbf{x}_B^* - t\mathbf{d}$, with t standing for the new value of the entering variable. This value t equals $\bar{b}_i/\bar{a}_{ij}$, which may be written as x_i^*/w_j.

To answer the fourth question, let us record the last row of (10.7) as

$$z = \bar{d} + \bar{c}_j x_j + \sum_{s \in R} \bar{c}_s x_s$$

with $s \in R$ if and only if $s \in N$ and $s \neq j$. Since pivoting amounts to substituting for x_j from

$$x_j = \left(\bar{b}_i - x_i - \sum_{s \in R} \bar{a}_{is} x_s\right)\Big/\bar{a}_{ij}$$

the formula for z gets updated into

$$z = \bar{d} + \bar{c}_j\left(\bar{b}_i - x_i - \sum_{s \in R} \bar{a}_{is} x_s\right)\Big/\bar{a}_{ij} + \sum_{s \in R} \bar{c}_s x_s.$$

After simplification, and writing w_s for each $\bar{a}_{is}(s \in N)$, we obtain

$$z = \left(\bar{d} + \bar{b}_i \frac{\bar{c}_j}{w_j}\right) - (\bar{c}_j/w_j)x_i + \sum_{s \in R}\left(\bar{c}_s - w_s \frac{\bar{c}_j}{w_j}\right)x_s.$$

Thus, the new coefficient $\bar{c}_i$ at x_i equals $-\bar{c}_j/w_j$, and the coefficient $\bar{c}_s$ at each x_s with $s \in R$ gets replaced by $\bar{c}_s - \bar{c}_j w_s/w_j = \bar{c}_s + \bar{c}_i w_s$.

These findings are summarized in Box 10.1. Rather than beginning with a dual-feasible dictionary (10.7), the iteration begins just with a vector $\mathbf{x}_B^*$ whose components $x_r^*(r \in B)$ are the numbers $\bar{b}_r$ and with a vector $\bar{\mathbf{c}}_N$ whose components are the numbers $\bar{c}_s(s \in N)$.

BOX 10.1 Iteration of the Revised Dual Simplex Method

Step 1. If $\mathbf{x}_B^* \geq 0$ then stop: $\mathbf{x}^*$ is an optimal solution. Otherwise, choose the leaving variable; this may be any basic variable x_i with $x_i^* < 0$.

Step 2. Solve the system $\mathbf{vB} = \mathbf{e}$ with $\mathbf{e}$ standing for the pth row of the identity matrix and with p such that x_i appears in the pth position of the basis heading. Compute $\mathbf{w}_N = \mathbf{vA}_N$.

Step 3. Let J be the set of those nonbasic variables x_j for which $w_j < 0$. If J is empty then stop: the problem is infeasible. Otherwise, find the x_j in J that minimizes $\bar{c}_j/w_j$ and let it be the entering variable.

Step 4. Solve the system $\mathbf{Bd} = \mathbf{a}$ with $\mathbf{a}$ standing for the entering column.

Step 5. Set the value x_j^* of the entering variable at $t = x_i^*/w_j$ and replace the values $\mathbf{x}_B^*$ of the basic variables by $\mathbf{x}_B^* - t\mathbf{d}$. Replace the leaving column of $\mathbf{B}$ by the entering column and, in the basis heading, replace the leaving variable by the entering variable. Set $\bar{c}_i = -\bar{c}_j/w_j$ and add $\bar{c}_i w_s$ to each $\bar{c}_s$ with $s \neq i$.

For illustration, we shall apply the revised dual simplex method to the problem

$$\text{minimize} \quad \mathbf{cx} \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

with

$$\mathbf{A} = \begin{bmatrix} -6 & 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & -2 & -1 & -5 & 0 & 1 & 0 \\ -2 & 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 14 \\ -25 \\ 14 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} -5 & -3 & -3 & -6 & 0 & 0 & 0 \end{bmatrix}.$$

We may initialize by

$$\mathbf{x}_B^* = \begin{bmatrix} x_5^* \\ x_6^* \\ x_7^* \end{bmatrix} = \begin{bmatrix} 14 \\ -25 \\ 14 \end{bmatrix}$$

and $\bar{\mathbf{c}}_N = [\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4] = [-5, -3, -3, -6]$.

The first iteration begins.

Step 1. The leaving variable is x_6.

Step 2. Since $\mathbf{B} = \mathbf{I}$, the system $\mathbf{vB} = [0, 1, 0]$ reduces to $\mathbf{v} = [0, 1, 0]$. We have

$$[w_1, w_2, w_3, w_4] = \mathbf{v} \begin{bmatrix} -6 & 1 & 2 & 4 \\ 3 & -2 & -1 & -5 \\ -2 & 1 & 0 & 2 \end{bmatrix} = [3, -2, -1, -5].$$

Step 3. The set J consists of x_2, x_3, x_4. Comparing the ratios 3/2, 3/1, and 6/5, we find that x_4 has to enter the basis.

Step 4. Since $\mathbf{B} = \mathbf{I}$, the system

$$\mathbf{Bd} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix} \quad \text{reduces to} \quad \mathbf{d} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$$

Step 5. We have $t = x_6^*/w_4 = 5$,

$$\mathbf{x}_B^* = \begin{bmatrix} x_5^* \\ x_4^* \\ x_7^* \end{bmatrix} = \begin{bmatrix} 14 - 4t \\ t \\ 14 - 2t \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}, \qquad \mathbf{B} = \mathbf{E}_1 \quad \text{with} \quad \mathbf{E}_1 = \begin{bmatrix} 1 & 4 & \\ & -5 & \\ & 2 & 1 \end{bmatrix}$$

$$\begin{aligned}[\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_6] &= [-5 + 3\bar{c}_6, -3 - 2\bar{c}_6, -3 - \bar{c}_6, \bar{c}_6] \\ &= [-8.6, -0.6, -1.8, -1.2].\end{aligned}$$

The second iteration begins.

Step 1. The leaving variable is x_5.

Step 2. Solving the system $\mathbf{vB} = [1, 0, 0]$ we find $\mathbf{v} = [1, 0.8, 0]$. Hence

$$[w_1, w_2, w_3, w_6] = \mathbf{v}\begin{bmatrix} -6 & 1 & 2 & 0 \\ 3 & -2 & -1 & 1 \\ -2 & 1 & 0 & 0 \end{bmatrix} = [-3.6, -0.6, 1.2, 0.8].$$

Step 3. The set J consists of x_1 and x_2. Comparing the ratios 8.6/3.6 and 0.6/0.6, we find that x_2 has to enter the basis.

Step 4. Solving the system

$$\mathbf{Bd} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{we find} \quad \mathbf{d} = \begin{bmatrix} -0.6 \\ 0.4 \\ 0.2 \end{bmatrix}.$$

Step 5. We have $t = x_5^*/w_2 = 10$,

$$x_B^* = \begin{bmatrix} x_2^* \\ x_4^* \\ x_7^* \end{bmatrix} = \begin{bmatrix} t \\ 5 - 0.4t \\ 4 - 0.2t \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{B} = \mathbf{E}_1\mathbf{E}_2 \quad \text{with} \quad \mathbf{E}_2 = \begin{bmatrix} -0.6 & & \\ 0.4 & 1 & \\ 0.2 & & 1 \end{bmatrix}$$

$$\begin{aligned}[\bar{c}_1, \bar{c}_3, \bar{c}_5, \bar{c}_6] &= [-8.6 - 3.6\bar{c}_5, -1.8 + 1.2\bar{c}_5, \bar{c}_5, -1.2 + 0.8\bar{c}_5] \\ &= [-5, -3, -1, -2].\end{aligned}$$

The third iteration begins.

Step 1. Since $\mathbf{x}_B^* \geq \mathbf{0}$, the current solution is optimal. (This observation could have been made in step 5 of the previous iteration; the update of $\overline{\mathbf{c}}_N$ was not strictly necessary.)

Extensions to the general setting of problems

$$\text{maximize} \quad \mathbf{cx} \qquad \text{subject to} \quad \mathbf{Ax} = \mathbf{b}, \quad \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \tag{10.12}$$

are as follows. For every basic solution of (10.12), as defined in Chapter 9, we have $\mathbf{cx} = \mathbf{c}_B\mathbf{x}_B + \mathbf{c}_N\mathbf{x}_N = \mathbf{c}_B(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{A}_N\mathbf{x}_N) + \mathbf{c}_N\mathbf{x}_N = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_N)\mathbf{x}_N$. We shall write $\overline{\mathbf{c}}_N = \mathbf{c}_N - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_N$. A basic solution $\mathbf{x}^*$ is *dual-feasible* if each nonbasic x_j with $\overline{c}_j > 0$ has $x_j^* = u_j$ and if each nonbasic x_j with $\overline{c}_j < 0$ has $x_j^* = l_j$. Each iteration of the revised dual simplex method, beginning with some dual-feasible basic solution $\mathbf{x}^*$ and with the corresponding vector $\mathbf{c}_N$, is as described in Box 10.2.

BOX 10.2 An Iteration of Revised Dual Simplex Method Extended

Step 1. If $\mathbf{l}_B \leq \mathbf{x}_B^* \leq \mathbf{u}_B$ then stop: $\mathbf{x}^*$ is an optimal solution. Otherwise, choose the leaving variable: this may be any basic variable x_i with $x_i^* < l_i$ or $x_i^* > u_i$.

Step 2. Solve the system $\mathbf{vB} = \mathbf{e}$ with $\mathbf{e}$ standing for the pth row of the identity matrix and with p such that x_i appears in the pth position of the basis heading. Compute $\mathbf{w}_N = \mathbf{vA}_N$.

Step 3. If $x_i^* < l_i$, then let J be the set of those nonbasic variables x_j for which $w_j < 0$, $x_j^* < u_j$ or $w_j > 0$, $x_j^* > l_j$. If $x_i^* > u_i$, then let J be the set of those nonbasic variables x_j for which $w_j > 0$, $x_j^* < u_j$ or $w_j < 0$, $x_j^* > l_j$. If J is empty then stop: the problem is infeasible. Otherwise, find the x_j in J that minimizes $|\overline{c}_j/w_j|$ and let it be the entering variable.

Step 4. Solve the system $\mathbf{Bd} = \mathbf{a}$ with $\mathbf{a}$ standing for the entering column.

Step 5. Set $t = (x_i^* - l_i)/w_j$ in case $x_i^* < l_i$ and $t = (x_i^* - u_i)/w_j$ in case $x_i^* > u_i$. Replace the value x_j^* of x_j by $x_j^* + t$ and replace the values $\mathbf{x}_B^*$ of the basic variables by $\mathbf{x}_B^* - t\mathbf{d}$. Replace the leaving column of $\mathbf{B}$ by the entering column and, in the basis heading, replace the leaving variable by the entering variable. Set $\overline{c}_i = -\overline{c}_j/w_j$ and add $\overline{c}_i w_s$ to each $\overline{c}_s$ with $s \neq i$.

We shall not worry about initializing the dual simplex method: whenever an application of this method is called for in sensitivity analysis, a dual-feasible basic solution is readily available.