Lecture 8 26.3 2019

Branch & Bound: A divide-and conquer strategy to solve any IP via LP-relaxation

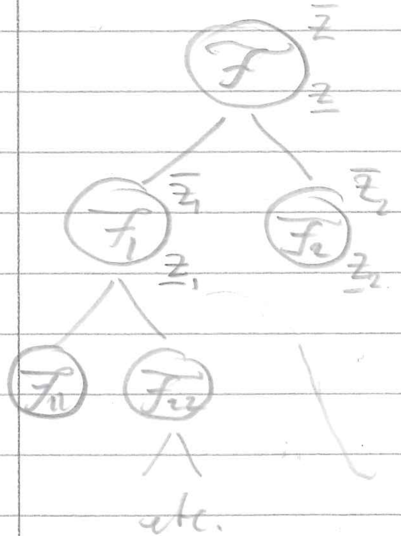
Notation: $(\mathcal{F})$ $z = \max\{c^T x : x \in \mathcal{P},\ x \in \mathbb{Z}^n\}$

for some formulation (polyhedron) $\mathcal{P}$ and feasible set $\mathcal{F} = \mathcal{P} \cap \mathbb{Z}^n$ and with LP-relaxation

$(\mathcal{P})$ $\bar{z} = \max\{c^T x : x \in \mathcal{P}\}$, yielding dual bound $\bar{z}$.

Also, $\underline{z}$ a primal bound (may be $-\infty$).

Idea: hierarchical splitting of $\mathcal{F}$



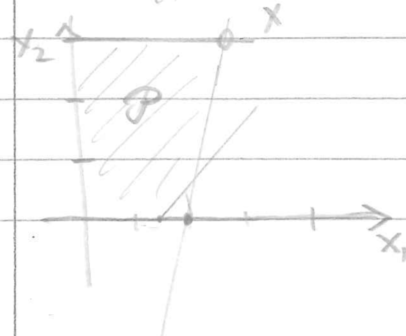
solve $(\mathcal{P}) \to \bar{z}$, possibly find $x \in \mathcal{F} \to \underline{z}$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, but $\mathcal{P}_1 \cup \mathcal{P}_2 \subsetneq \mathcal{P}$

$\mathcal{F}_{11} \cup \mathcal{F}_{12} = \mathcal{F}_1$, but $\mathcal{P}_{11} \cup \mathcal{P}_{12} \subsetneq \mathcal{P}_1$

if $\underline{z}_1 \geq \bar{z}_2$, no need to split $\mathcal{F}_2$

Example 1: $(\mathcal{F})$ $z = \max\{c^T x : Ax \leq b,\ x \geq 0,\ x \in \mathbb{Z}^2\}$,



where $c = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $A = \begin{bmatrix} 7 & -2 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 14 \\ 3 \\ 3 \end{bmatrix}$

(2)

Solve LP-relaxation. Simplex yields tableau

x_1	x_2	x_3	x_4	x_5	
1	0	1/7	2/7	0	20/7
0	1	0	1	0	3
0	0	-2/7	10/7	1	23/7
0	0	-4/7	-1/7	0	-59/7

$\Rightarrow \bar{x} = \left(\frac{20}{7}, 3\right)$

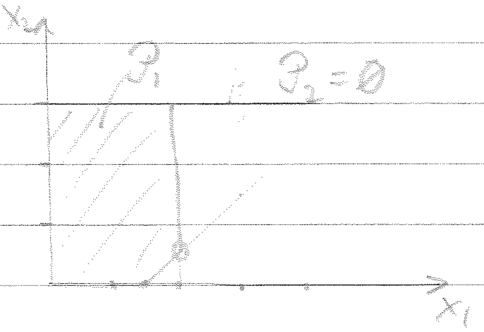
$\bar{z} = \left\lfloor \frac{59}{7} \right\rfloor = 8$ since $c \in \mathbb{Z}^2$

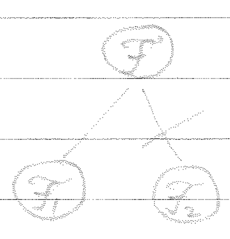
$\underline{z} = -\infty$

$\underline{z} < \bar{z} \Rightarrow (\mathcal{F})$ not solved to optimality.

Fractional Branching: $\bar{x}_1 = \frac{20}{7}$ not integer

$\Rightarrow \mathcal{P}_1 = \mathcal{P} \cap \{x : x_1 \leq \lfloor \frac{20}{7} \rfloor = 2\}$, $\mathcal{P}_2 = \mathcal{P} \cap \{x : x_1 \geq 3\}$





prune by infeasibility
$\mathcal{P}_2 = \emptyset \Rightarrow \mathcal{F}_2 = \emptyset$.

$(\mathcal{F}_1)$ $\max\{4x_1 - x_2 : 7x_1 - 2x_2 \leq 14,\ x_2 \leq 3,\ 2x_1 - 2x_2 \leq 3,$
$x_1 \leq 2,\ x \in \mathbb{Z}^2_+\}$.

$\mathcal{P}_1 \subsetneq \mathcal{P}$ and $\bar{x} \notin \mathcal{P}_1 \Rightarrow \bar{z}_1 = \max\{c^T x : x \in \mathcal{P}_1\} < \bar{z}$.

$(\mathcal{P})$ $\bar{x} = \arg\max c^T x$
s.t. $Ax \leq b$
$x \geq 0$

$(\mathcal{D})$ $\bar{y} = \arg\min b^T y$
s.t. $A^T y \geq c$
$y \geq 0$

$\downarrow$ $\qquad$ $\downarrow$

$(\mathcal{P}_1)$ $\max c^T x$
s.t. $Ax \leq b$
$x_1 \leq 2$
$x \geq 0$

$\bar{x}$ not feasible

$(\mathcal{D}_1)$ $\min b^T y$
s.t. $A^T y + y_4 \geq c$
$y, y_4 \geq 0$

$(\bar{y}, 0)$ feasible!

③

Dual Simplex Step

slack variable for new constraint

	x_1	x_2	x_3	x_4	x_5	x_6	
	1	0	4/7	2/7	0	0	20/7
	0	1	0	1	0	0	3
	0	0	-2/7	10/7	1	0	23/7
new → constr.	1	0	0	0	0	1	2
	0	0	-4/7	-1/7	0	0	-59/7

interpret as pivot variable having been increased too much → negative!

→ 0 0 $-\frac{4}{7}$ $-\frac{2}{7}$ 0 1 $-\frac{6}{7}$

eliminate

would like to keep negative … take one of these vars instead of x_6

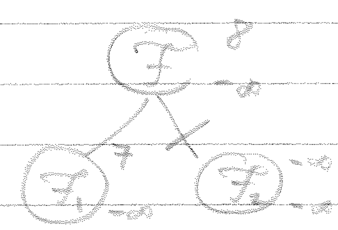
$\Rightarrow \min\left\{\frac{4/7}{4/7}, \frac{1/7}{2/7}\right\} = \frac{1}{2} \Rightarrow$ eliminate x_4

	1	0	0	0	0	1	2
	0	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	0
→	0	0	-1	0	1	5	-1
	0	0	$\frac{1}{2}$	1	0	$-\frac{3}{2}$	3
	0	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-8

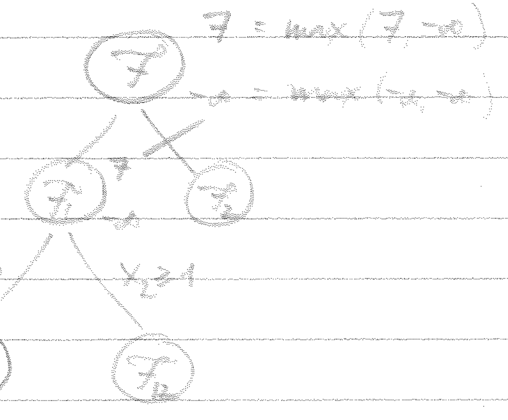
1	0	0	0	0	1	2
0	1	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$
0	0	1	0	-1	-5	1
0	0	0	1	$\frac{1}{2}$	-1	$\frac{7}{2}$
0	0	0	0	$-\frac{1}{2}$	$-\frac{6}{2}$	$-\frac{15}{2}$

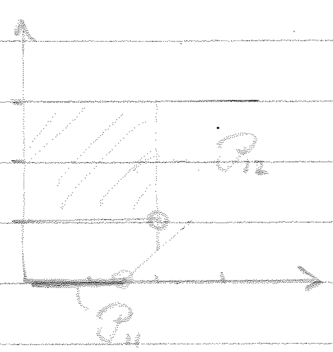
re-optimized!

$x^{(1)} = \left(2, \frac{1}{2}\right), \quad z^{(1)} = \left\lfloor \frac{15}{2} \right\rfloor = 7$

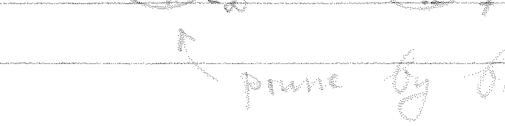


propagate → bounds





Apply dual simplex to P_{11}, P_{12}



prune by bound!

④

General principles:

Proposition 1: Let $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ and

$z^{(j)} = \max\{c^T x : x \in \mathcal{F}_j\}$. Then $z = \max_j z^{(j)}$.

Proof: $\forall j,\ \mathcal{F}_j \subset \mathcal{F} \Rightarrow z^{(j)} \le z$.

Let $x^* = \operatorname{argmax}\{c^T x : x \in \mathcal{F}\}$. Then $\exists\, i$ s.t.

$x^* \in \mathcal{F}_i$ and $z = c^T x^* \le z^{(i)}$ $\square$.

Proposition 2: Let $\underline{z}^{(j)} \le z^{(j)} \le \bar{z}^{(j)}$ $(j = 1, \ldots, k)$.

Then $\max_j \underline{z}^{(j)} \le z \le \max_j \bar{z}^{(j)}$.

Proof: $\underline{z}^{(j)} \le z^{(j)} \le z \ \forall j \Rightarrow \max_j \underline{z}^{(j)} \le z$.

Moreover, $z = \max_j z^{(j)} \le \max_j \bar{z}^{(j)}$. $\square$

Proposition 3: Let $\underline{z} \le z \le \bar{z}$ and $\underline{z}^{(j)} \le z^{(j)} \le \bar{z}^{(j)}$

$\forall j$. If $\bar{z}^{(j)} < \underline{z}$ then $x^* \notin \mathcal{F}_j$.

Proof: $\forall x \in \mathcal{F}_j$, $c^T x \le z^{(j)} \le \bar{z}^{(j)} < \underline{z} \le z = c^T x^*$

$\square$