Numerical Solution of Differential Equations: Problem Sheet 2 (of 6)

1. Consider the initial-value problem

$$y' = \log\log(4 + y^2), \qquad x \in [0, 1], \qquad y(0) = 1,$$

and the sequence $(y_n)_{n=0}^{N}$, $N \geq 1$, generated by the explicit Euler method

$$y_{n+1} = y_n + h\log\log(4 + y_n^2), \qquad n = 0, \ldots, N-1, \qquad y_0 = 1,$$

using the mesh points $x_n = nh$, $n = 0, \ldots, N$, with spacing $h = 1/N$. Here log denotes the logarithm with base e.

(a) Let T_n denote the consistency error of Euler's method at $x = x_n$ for this initial value problem. Show that $|T_n| \leq h/(4\mathrm{e})$.

b) Verify that

$$|y(x_{n+1}) - y_{n+1}| \leq (1 + hL)|y(x_n) - y_n| + h|T_n|, \qquad n = 0, \ldots, N-1,$$

where $L = 1/(2\log 4)$.

c) Find a positive integer N_0, as small as possible, such that

$$\max_{0 \leq n \leq N} |y(x_n) - y_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

2. The explicit Euler method, the implicit Euler method, and the implicit midpoint rule are Runge–Kutta methods. Write down the formulae for their stages when considered as Runge–Kutta methods.

3. Consider the Runge–Kutta method $y_{n+1} = y_n + h(\alpha\, k_1 + \beta\, k_2)$ where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + \gamma h, y_n + \gamma h k_1)$, and where α, β, γ are real parameters.

(a) Show that there is a choice of these parameters such that the order of the method is 2.

(b) Suppose that a second-order method of the above form is applied to the initial value problem $y' = -\lambda y$, $y(0) = 1$, where λ is a positive real number. Show that the sequence $(y_n)_{n\geq 0}$ is bounded if and only if $h \leq \frac{2}{\lambda}$.
Show further that, for such λ,

$$|y(x_n) - y_n| \leq \frac{1}{6}\lambda^3 h^2 x_n, \quad n \geq 0.$$

4. a) What does it mean to say that a linear multistep method is *zero-stable*? Formulate an equivalent characterization of zero-stability of a linear multistep method in terms of the roots of its first characteristic polynomial.

b) Define the consistency error of a linear multistep method.

c) Show that there is a value of the parameter b such that the linear multistep method defined by the formula $y_{n+3} + (2b-3)(y_{n+2} - y_{n+1}) - y_n = hb(f_{n+2} + f_{n+1})$ is fourth-order accurate. Show further that the method is *not* zero–stable for this value of b.