# Numerical Solution of Differential Equations: Problem Sheet 4 (of 6)

1. We consider the system of scalar ODEs
$$y' = v, \quad v' = f(y), \tag{1}$$
where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.

   (a) Let $F$ be a primitive function of $f$. Show that $H(v, y) = v^2/2 - F(y)$ is a Hamiltonian of (1) and verify that it is indeed a first integral.

   (b) Let $\boldsymbol{z} = \begin{pmatrix} y \\ v \end{pmatrix}$ and $\boldsymbol{g}(\boldsymbol{z}) = \begin{pmatrix} v \\ f(y) \end{pmatrix}$, and let $\boldsymbol{\Psi}$ be the discrete evolution operator of the implicit midpoint rule associated with (1). Show that
$$\boldsymbol{D}_{\boldsymbol{z}_0}(\boldsymbol{\Psi}(0, \boldsymbol{z}_0, h, \boldsymbol{g})) = \frac{1}{1 - \frac{h^2}{4} f'(*)} \begin{pmatrix} 1 + \frac{h^2}{4} f'(*) & h \\ h f'(*) & 1 + \frac{h^2}{4} f'(*) \end{pmatrix},$$
where $f'(*) := f'\left(\frac{y_0+y_1}{2}\right)$.

   (c) Hence deduce that the implicit midpoint rule is symplectic.

Suppose that we have discrete data $\{U_j\}$ defined on an infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$. Let $\delta$ and $\mu$ be the discrete differentiation and smoothing operators defined by
$$(\delta U)_j = (U_{j+1} - U_{j-1})/(2\Delta x), \qquad (\mu U)_j = (U_{j+1} + U_{j-1})/2.$$

2. Determine the functions $\delta U$, $\delta V$, $\mu U$, $\mu V$ for $U = (\ldots, 1, -1, 1, -1, 1, -1, 1, \ldots)$ and $V = (\ldots, 1, 0, -1, 0, 1, 0, -1, 0, \ldots)$.

3. Determine what effect $\delta$ and $\mu$ have on the function $U$ defined by $U_j = \mathrm{e}^{\imath k x_j}$, $j = 0, \pm 1, \pm 2, \ldots$, where $k$ is a real constant (the wave number).

4. The semidiscrete Fourier transform of a function $U$ defined on the infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$, is the function $k \mapsto \hat{U}(k)$, $k \in [-\pi/\Delta x, \pi/\Delta x]$, defined by
$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} U_j \,.$$
[The reason for the restriction on $k$ is that the wave numbers $|k| > \pi/\Delta x$ are not resolvable on a grid of spacing $\Delta x$; this is the phenomenon of *aliasing*.]

   Show that the inverse of the semidiscrete Fourier transform is given by the formula
$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath k j \Delta x} \hat{U}(k) \, \mathrm{d}k \,.$$

   Describe the relationship between $\hat{U}(k)$, and $\widehat{\delta U}(k)$ and $\widehat{\mu U}(k)$. [Note that this is a restatement of Question 3.]

P.T.O.

The ratios $\widehat{\delta U}/\hat{U}$ and $\widehat{\mu U}/\hat{U}$ are referred to as *Fourier multipliers*. Sketch the graphs of these Fourier multipliers as functions of $k \in [-\pi/\Delta x, \pi/\Delta x]$.

One would think that applying $\mu$ repeatedly to $U$ should lead to a function that is much smoother than $U$. Explain this effect by considering a sketch of the multiplier function $\widehat{\mu^m U}/\hat{U}$ for $m \gg 1$. Your analysis should reveal that taking successive powers of $\mu$ is not a perfect smoothing procedure. Explain.