Numerical Solution of Differential Equations: Problem Sheet 5 (of 6)

1. The $\ell_2(-\infty,\infty)$ norm of U and the $L_2(-\pi/\Delta x,\, \pi/\Delta x)$ norm of $\hat{U}$ are defined, respectively, by

$$\|U\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j|^2\right)^{1/2}, \qquad \|\hat{U}\|_{L_2} = \left(\int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \,\mathrm{d}k\right)^{1/2}.$$

Prove *Parseval's identity*:

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}}\|\hat{U}\|_{L_2}.$$

2. In the lectures we considered the simplest finite difference approximation of the heat equation $u_t = u_{xx}$, given by

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}, \qquad j = \ldots,-2,-1,0,1,2,\ldots\,; \quad n = 0,1,2,\ldots.$$

What would the analogous difference approximation be based on values of U at just every other point in the x direction, i.e., U_{j+2}^n, U_j^n and U_{j-2}^n? Now suppose that you create a new difference approximation from these two schemes by adding 1/2 of the first difference approximation to 1/2 of the second difference approximation. Using Fourier analysis, explore how large Δt can be in relation to Δx if this last scheme is to be stable in the norm of $\ell_2 = \ell_2(-\infty,\infty)$.

3. Consider the implicit Euler scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + b\,\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = a\,\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}, \qquad j = 0,\pm1,\pm2,\ldots, \quad n \geq 0,$$

$$U_j^0 = u_0(x_j), \qquad j = 0,\pm1,\pm2,\ldots,$$

for the numerical solution of the initial-value problem

$$\frac{\partial u}{\partial t} + b\,\frac{\partial u}{\partial x} = a\,\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty,$$

where $a > 0$ and b are fixed real numbers. Show that the scheme is unconditionally stable in the ℓ_2 norm.

Show further that the consistency error $|T_j^n| \leq C(\Delta t + (\Delta x)^2)$ for all $n \geq 0$ and $j = 0,\pm1,\pm2,\ldots$, where C is a constant independent of Δt and Δx, provided that $\partial^2 u/\partial t^2$, $\partial^3 u/\partial x^3$ and $\partial^4 u/\partial x^4$ exist and are bounded functions of x and t, $(x,t) \in (-\infty,\infty)\times[0,\infty)$.

4. Consider the θ-method for the numerical solution of the initial-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty.$$

Suppose that the parameter θ has been chosen according to the formula

$$\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t}.$$

Show that the resulting scheme is unconditionally stable in the ℓ_2 norm and has a consistency error which is $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$, provided that derivatives of u of sufficiently high order exist and are bounded functions of x and t, $(x, t) \in (-\infty, \infty) \times [0, \infty)$.