Numerical Solution of Differential Equations? Why?

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Numerical solution of differential equations is a rich and active field of modern applied mathematics.

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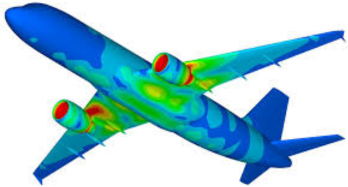
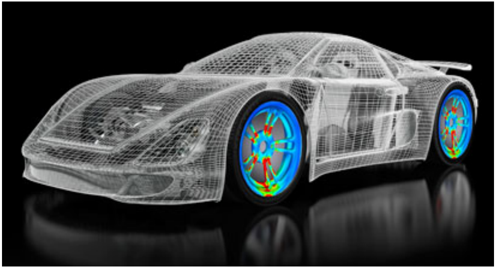
The growth of the subject is stimulated by ever-increasing demands from the natural sciences, engineering and economics to provide accurate and reliable approximations to mathematical models involving ODEs & PDEs whose exact solutions are either too complicated to determine in closed form or are not known to exist.





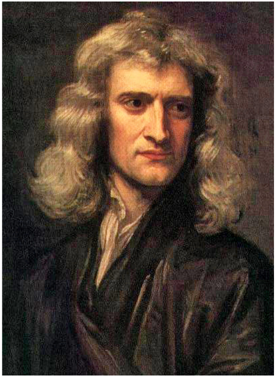
Evolution of crack-fields

Numerical approximation of the Francfort–Marigo model of brittle fracture.

Evolution of the computational grids

Numerical approximation of the Francfort–Marigo model of brittle fracture.

The foundations of the theory of differential equations were laid by Leibniz, the Bernoulli brothers, and others from the 1680s, not long after Newton introduced his 'fluxional equations' in the 1670s.



Sir Isaac Newton
1643–1727

Gottfried Wilhelm von Leibniz
1646–1716

Around 1671, Newton wrote his, then unpublished, *The Method of Fluxions and Infinite Series* (published in 1736), in which he classified first-order differential equations, known to him as fluxional equations, into three classes, as follows (using modern notation):

$$\underbrace{\frac{\mathrm{d}y}{\mathrm{d}x} = f(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),}_{\text{Ordinary Differential Equations (ODEs)}}$$

$$\underbrace{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.}_{\text{Partial Differential Equation (PDE)}}$$

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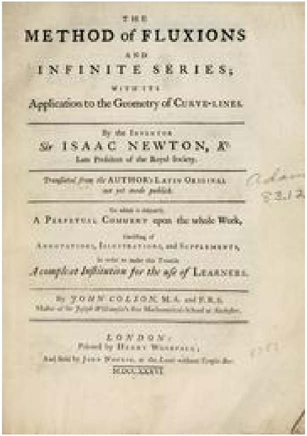
In the same year, Leibniz introduced the term *differential equation* (aequatio differentialis, Latin).



THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, K^t

Late President of the Royal Society.

Translated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

A PERPETUAL COMMENT upon the whole Work,

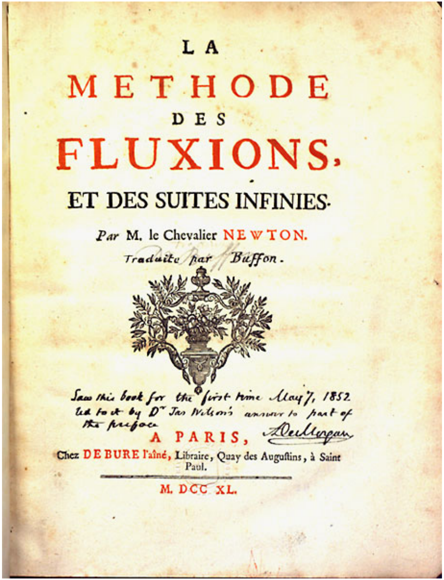
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

A compleat Institution for the use of LEARNERS.

By JOHN COLSON, M.A. and F.R.S.

LONDON:

MDCCXXXVI



LA

METHODE

DES

FLUXIONS,

ET DES SUITES INFINIES.

Par M. le Chevalier NEWTON.

Traduite par Buffon.

Saw this book for the first time May 7, 1852
led to it by Dr Jas Wilson's answer to part of
the preface

A PARIS,

A De Morgan

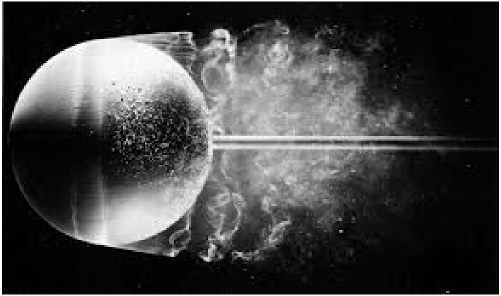
Chez DE BURE l'aîné, Libraire, Quay des Augustins, à Saint Paul.

M. DCC XL.

Many differential equations are (much!) more complicated

Many differential equations are (much!) more complicated

Sphere in a turbulent flow:



Source:

Milton Van Dyke, An Album of Fluid Motion, Parabolic Press, 12th ed., 1982.

Many differential equations are (much!) more complicated

Many differential equations are (much!) more complicated

The compressible Navier–Stokes equations:

Suppose that $\Omega \subset \mathbb{R}^3$. Given

$$\rho_0 = \rho_0(\mathbf{x}), \qquad \mathbf{u}_0 = \mathbf{u}(\mathbf{x}), \qquad \mathbf{f} = \mathbf{f}(\mathbf{x}, t),$$

find

$$\rho = \rho(\mathbf{x}, t), \qquad \mathbf{u} = \mathbf{u}(\mathbf{x}, t),$$

such that:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\,\rho) &= 0 && \text{in } \Omega \times (0,\infty),\\
\rho(\mathbf{x},0) &= \rho_0(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega,\\
\frac{\partial(\rho\,\mathbf{u})}{\partial t} + \nabla \cdot (\rho\,\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbf{S}(\mathbf{u},\rho) + \nabla\, p(\rho) &= \rho\,\mathbf{f} && \text{in } \Omega \times (0,\infty),\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times (0,\infty),\\
(\rho\,\mathbf{u})(\mathbf{x},0) &= (\rho_0\,\mathbf{u}_0)(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega.
\end{aligned}$$

P.-L. Lions (OUP, 1996, 1998).

Mathematics of numerical algorithms?

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The sinking of the Sleipner A offshore platform in Gandsfjorden near Stavanger, Norway, on August 23, 1991, resulted in a loss of nearly one billion dollars.

It was found to be the result of inaccurate numerical simulation.



A landmark contribution to the foundations of the mathematical theory of numerical methods for PDEs is *Über die partiellen Differenzengleichungen der mathematischen Physik* by Richard Courant, Karl Friedrichs, and Hans Lewy, (Mathematische Annalen, 1928).

The basic idea

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... is extremely simple. Suppose that y is differentiable at $x \in \mathbb{R}$; then,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

Thus,

$$\frac{y(x+h) - y(x)}{h} = y'(x) + o(1) \qquad \text{as } h \to 0.$$

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Analogously, if y' is differentiable, then

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \qquad \text{as } h \to 0.$$

Euler's method

Leonhard Euler (1707–1783)

Euler's method for $y'(x) = f(x, y(x))$ subject to the i.c. $y(x_0) = y_0$:

$$\frac{y(x_k + h) - y(x_k)}{h} \approx f(x_k, y(x_k)), \qquad y(x_0) = y_0, \qquad x_k = x_0 + kh,$$

for $k = 0, 1, \ldots$.

Cosmological simulation of the evolution of the Universe

Click here

Volker Springler (Astrophysics, U. of Heidelberg)
Millennium-XXL project: $6720^3 \approx 303 \times 10^9$ particles over the equivalent of more than 13×10^9 years. Largest N-body simulation ever: required the equivalent of 300 years of CPU time and used more than 12000 computer cores and 30 TB of RAM on the Juropa Machine at the Jülich Supercomputer Centre in Germany.