Numerical Solution of Differential Equations I

Endre Süli

Mathematical Institute
University of Oxford

2019

Lecture 1

Picard's Theorem

Ordinary differential equations frequently occur as mathematical models in science, engineering, and economics. It is seldom that these equations have solutions that can be expressed in closed form, so it is common to seek approximate solutions.

We shall be concerned with the construction and the analysis of numerical methods for first-order differential equations of the form

$$y' = f(x, y) \tag{1}$$

for the real-valued function y of $x \in \mathbb{R}$, where $y' \equiv \mathrm{d}y/\mathrm{d}x$.

The ODE (1) will be considered together with an **initial condition**: given two real numbers x_0 and y_0, find a solution to (1) for $x > x_0$ such that

$$y(x_0) = y_0. \tag{2}$$

The problem (1), (2) is called an **initial-value problem**.

Theorem (Picard's Theorem[1])

Suppose that $f(\cdot,\cdot)$ is a continuous function of its arguments in a region U of the (x, y) plane which contains the rectangle

$$\mathrm{R} := \{(x, y) \,:\, x_0 \leq x \leq X_M, \quad |y - y_0| \leq Y_M\},$$

where $X_M > x_0$ and $Y_M > 0$ are constants.
Suppose also, that there exists a positive constant L such that

$$|f(x, y) - f(x, z)| \leq L|y - z| \tag{3}$$

holds whenever (x, y) and (x, z) lie in the rectangle R*. Letting*

$$M := \max\{|f(x, y)| \,:\, (x, y) \in \mathrm{R}\},$$

suppose that $M(X_M - x_0) \leq Y_M$. Then, there exists a unique continuously differentiable function $x \mapsto y(x)$, defined on the closed interval $[x_0, X_M]$, which satisfies (1) and (2).

---

[1] Emile Picard (1856–1941)

The condition (3) is called a **Lipschitz condition**[2], and L is called a **Lipschitz constant** for f. The proof is based on considering the sequence of functions $\{y_n\}_{n=0}^{\infty}$, defined by the *Picard Iteration*:

$$\begin{aligned} y_0(x) &\equiv y_0, \\ y_n(x) &:= y_0 + \int_{x_0}^{x} f(\xi, y_{n-1}(\xi))\,\mathrm{d}\xi, \qquad n = 1, 2, \ldots, \end{aligned} \tag{4}$$

and show, using the conditions of the theorem, that $\{y_n\}_{n=0}^{\infty}$, as a sequence of continuous functions, converges uniformly on the interval $[x_0, X_M]$ to a continuous function y defined on $[x_0, X_M]$ s.t.

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))\,\mathrm{d}\xi.$$

This implies that y is continuously differentiable on $[x_0, X_M]$ and it satisfies the differential equation (1) and the initial condition (2). The uniqueness of the solution follows from the Lipschitz condition.

[2] Rudolf Lipschitz (1832–1903)

Picard's Theorem has a natural extension to an initial-value problem for a system of m differential equations of the form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(x_0) = \mathbf{y}_0, \tag{5}$$

where $\mathbf{y}_0 \in \mathbb{R}^m$ and $\mathbf{f} : [x_0, X_M] \times \mathbb{R}^m \to \mathbb{R}^m$.

Consider the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^m$ defined by

$$\|v\| := \left(\sum_{i=1}^{m} |v_i|^2 \right)^{1/2}, \qquad v \in \mathbb{R}^m.$$

We then have the following result.

Theorem (Picard's Theorem for systems)

Suppose that $\mathbf{f}(\cdot,\cdot)$ *is a continuous function of its arguments in a region* U *of the* $(x,\mathbf{y})$ *space* $\mathbf{R}^{1+m}$ *containing the parallelepiped*

$$\mathrm{R} := \{(x,\mathbf{y}) \,:\, x_0 \le x \le X_M, \quad \|\mathbf{y}-\mathbf{y}_0\| \le Y_M\},$$

where $X_M > x_0$ *and* $Y_M > 0$ *are constants.*
Suppose also that there exists a positive constant L *such that*

$$\|\mathbf{f}(x,\mathbf{y}) - \mathbf{f}(x,\mathbf{z})\| \le L\|\mathbf{y}-\mathbf{z}\| \tag{6}$$

holds whenever $(x,\mathbf{y})$ *and* $(x,\mathbf{z})$ *lie in* R*. Finally, letting*

$$M := \max\{\|\mathbf{f}(x,\mathbf{y})\| \,:\, (x,\mathbf{y}) \in \mathrm{R}\},$$

suppose that $M(X_M - x_0) \le Y_M$*. Then, there exists a unique continuously differentiable function* $x \mapsto \mathbf{y}(x)$*, defined on the closed interval* $[x_0, X_M]$*, which satisfies (5).*

We conclude by introducing the notion of *stability*.

Definition

A solution $\mathbf{y} = \mathbf{v}(x)$ to (5) is said to be **stable** on the interval $[x_0, X_M]$ if:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall \mathbf{z} \quad \text{satisfying} \quad \|\mathbf{v}(x_0) - \mathbf{z}\| < \delta$$

the solution $\mathbf{y} = \mathbf{w}(x)$ to the differential equation $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ satisfying the initial condition $\mathbf{w}(x_0) = \mathbf{z}$ is defined for all $x \in [x_0, X_M]$ and satisfies

$$\|\mathbf{v}(x) - \mathbf{w}(x)\| < \varepsilon \qquad \text{for all } x \text{ in } [x_0, X_M].$$

Using this definition, we can state the following theorem.

Theorem

Under the hypotheses of Picard's Theorem, the (unique) solution $\mathbf{y} = \mathbf{v}(x)$ *to the initial-value problem (5) is stable on the interval* $[x_0, X_M]$, *(where we assume that* $-\infty < x_0 < X_M < \infty$*).*

PROOF: Since

$$\mathbf{v}(x) = \mathbf{v}(x_0) + \int_{x_0}^{x} \mathbf{f}(\xi, \mathbf{v}(\xi))\,\mathrm{d}\xi$$

and

$$\mathbf{w}(x) = \mathbf{z} + \int_{x_0}^{x} \mathbf{f}(\xi, \mathbf{w}(\xi))\,\mathrm{d}\xi,$$

it follows that

$$\begin{aligned} \|\mathbf{v}(x) - \mathbf{w}(x)\| &\leq \|\mathbf{v}(x_0) - \mathbf{z}\| + \int_{x_0}^{x} \|\mathbf{f}(\xi, \mathbf{v}(\xi)) - \mathbf{f}(\xi, \mathbf{w}(\xi))\|\,\mathrm{d}\xi \\ &\leq \|\mathbf{v}(x_0) - \mathbf{z}\| + L \int_{x_0}^{x} \|\mathbf{v}(\xi) - \mathbf{w}(\xi)\|\,\mathrm{d}\xi. \qquad (7) \end{aligned}$$

Now put $A(x) := \|\mathbf{v}(x) - \mathbf{w}(x)\|$ and $a := \|\mathbf{v}(x_0) - \mathbf{z}\|$; then, (7) can be written as

$$A(x) \leq a + L \int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi, \qquad x_0 \leq x \leq X_M. \qquad (8)$$

Multiplying (8) by $\exp(-Lx)$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-Lx}\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi\right] \leq a\mathrm{e}^{-Lx}. \tag{9}$$

Integrating the inequality (9), we deduce that

$$\mathrm{e}^{-Lx}\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi \leq \frac{a}{L}\left(\mathrm{e}^{-Lx_0} - \mathrm{e}^{-Lx}\right),$$

that is

$$L\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi \leq a\left(\mathrm{e}^{L(x-x_0)} - 1\right). \tag{10}$$

Now substituting (10) into (8) gives

$$A(x) \leq a\mathrm{e}^{L(x-x_0)}, \qquad x_0 \leq x \leq X_M. \tag{11}$$

[The implication "(8) $\Rightarrow$ (11)" is called **Gronwall's Lemma**.]

Returning to our original notation, we deduce from (11) that

$$\|\mathbf{v}(x) - \mathbf{w}(x)\| \leq \|\mathbf{v}(x_0) - \mathbf{z}\| \, \mathrm{e}^{L(x-x_0)}, \qquad x_0 \leq x \leq X_M. \quad (12)$$

Thus, given $\varepsilon > 0$ as in Definition 3, we choose

$$\delta = \varepsilon \exp(-L(X_M - x_0))$$

to deduce from (12) stability. $\diamond$

Remark

A solution which is stable on $[x_0, \infty)$ *(i.e. stable on* $[x_0, X_M]$ *for each* X_M *and with* δ *independent of* X_M*) is said to be* **stable in the sense of Lyapunov**.

Moreover, if

$$\lim_{x \to \infty} \|\mathbf{v}(x) - \mathbf{w}(x)\| = 0,$$

then the solution $\mathbf{y} = \mathbf{v}(x)$ *is called* **asymptotically stable**.