Numerical Solution of Differential Equations I

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Lecture 3

General one-step methods

Definition

A **one-step method** is a function Ψ that takes the triplet

$$(\xi, \eta; h) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$$

and a function f, and computes an approximation $\Psi(\xi, \eta; h, f) \in \mathbb{R}$ of $y(\xi + h)$, which is the solution at $x = \xi + h$ of the initial-value problem

$$y'(x) = f(x, y(x)), \qquad y(\xi) = \eta.$$

The step size h may need to be assumed to be sufficiently small for Ψ to be well-defined.

Example

In the case of the implicit Euler method the function Ψ is defined implicitly, by

$$\Psi(\xi, \eta; h, f) = \eta + hf(\xi + h, \Psi(\xi, \eta; h, f)).$$

Let f satisfy the Lipschitz condition with Lipschitz constant L.

The Contraction Mapping Theorem implies that, given a pair $(\xi, \eta) \in \mathbb{R}^2$ and $h \in (0, 1/L)$, there exists a unique $\Psi(\xi, \eta; h, f)$ in $\mathbb{R}$ satisfying this implicit relationship.

Therefore, for such a "sufficiently small" h, the function Ψ associated with the implicit Euler method is well-defined.

Example

In the case of a general explicit one-step methods

$$\Psi(\xi, \eta; h, f) = \eta + h\,\Phi(\xi, \eta; h, f),$$

where $\Phi(\xi, \eta; h, f)$ can be explicitly computed without solving implicit equations in terms of ξ, η, h, and f.

In what follows, we shall not indicate the dependence of $\Phi(\xi, \eta; h, f)$ on f, and will write $\Phi(\xi, \eta; h)$ instead.

Example

In the case of the explicit Euler method:

$$\Psi(\xi, \eta; h, f) = \eta + hf(\xi, \eta).$$

General explicit one-step method

A general explicit one-step method may be written in the form:

$$y_{n+1} = y_n + h\,\Phi(x_n, y_n; h), \quad n = 0, \ldots, N-1, \quad y_0 = y(x_0)\,[= \text{given}],$$

where $\Phi(\cdot, \cdot; \cdot)$ is a continuous function of its variables.

Example

For the improved Euler method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

we have that

$$\Phi(\xi, \eta; h) = \frac{1}{2}\left[f(\xi, \eta) + f(\xi + h, \eta + hf(\xi, \eta))\right].$$

In order to assess the accuracy of a one-step method we define the **global error**, e_n, by

$$e_n = y(x_n) - y_n.$$

We define the **consistency error**, T_n, of the method by

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h).$$

Theorem

Consider a general explicit one-step method where, in addition to being a continuous function of its arguments, Φ is assumed to satisfy a Lipschitz condition with respect to its second argument; namely, there exists a positive constant L_Φ such that, for $0 \leq h \leq h_0$ and for the same region R *as in Picard's Theorem,*

$$|\Phi(x, y; h) - \Phi(x, z; h)| \leq L_\Phi |y - z|, \qquad \text{for } (x, y),\ (x, z) \text{ in } \mathrm{R}.$$

Then, assuming that $|y_n - y_0| \leq Y_M$, it follows that

$$|e_n| \leq \mathrm{e}^{L_\Phi (x_n - x_0)} |e_0| + \left[\frac{\mathrm{e}^{L_\Phi (x_n - x_0)} - 1}{L_\Phi} \right] T, \quad n = 0, \ldots, N,$$

where $T = \max_{0 \leq n \leq N-1} |T_n|$.

PROOF: Clearly,

$$e_{n+1} = e_n + h[\Phi(x_n, y(x_n); h) - \Phi(x_n, y_n; h)] + hT_n.$$

As $(x_n, y(x_n)), (x_n, y_n) \in \mathrm{R}$, the Lipschitz condition implies

$$|e_{n+1}| \le |e_n| + hL_\Phi |e_n| + h|T_n|, \qquad n = 0, \ldots, N-1.$$

That is,

$$|e_{n+1}| \le (1 + hL_\Phi)|e_n| + h|T_n|, \qquad n = 0, \ldots, N-1.$$

Hence,

$$\begin{aligned}
|e_1| &\le (1 + hL_\Phi)|e_0| + hT, \\
|e_2| &\le (1 + hL_\Phi)^2|e_0| + h[1 + (1 + hL_\Phi)]T, \\
|e_3| &\le (1 + hL_\Phi)^3|e_0| + h[1 + (1 + hL_\Phi) + (1 + hL_\Phi)^2]T, \\
&\text{etc.} \\
|e_n| &\le (1 + hL_\Phi)^n|e_0| + [(1 + hL_\Phi)^n - 1]T/L_\Phi.
\end{aligned}$$

As $1 + hL_\Phi \le \exp(hL_\Phi)$, we obtain the stated bound. $\diamond$

Example

Consider the initial-value problem $y' = \tan^{-1} y$, $y(0) = y_0$, and suppose that this is solved by the explicit Euler method.

The aim of the exercise is to quantify the size of the associated global error; thus, we need to find L and M_2.

Here $f(x, y) = \tan^{-1} y$, so by the Mean-Value Theorem

$$|f(x, y) - f(x, z)| = \left| \frac{\partial f}{\partial y}(x, \eta)\,(y - z) \right|,$$

where η lies between y and z. In our case

$$\left| \frac{\partial f}{\partial y} \right| = |(1 + y^2)^{-1}| \leq 1,$$

and therefore $L = 1$.

To find M_2 we need to obtain a bound on $|y''|$ (without actually solving the initial-value problem). By differentiating both sides of the differential equation with respect to x:

$$y'' = \frac{d}{dx}(\tan^{-1} y) = (1 + y^2)^{-1}\frac{dy}{dx} = (1 + y^2)^{-1}\tan^{-1} y.$$

Therefore $|y''(x)| \leq M_2 = \frac{1}{2}\pi$. Inserting the values of L and M_2 into the error bound from the last theorem:

$$|e_n| \leq \mathrm{e}^{x_n}|e_0| + \frac{1}{4}\pi\left(e^{x_n} - 1\right)h, \quad n = 0, \ldots, N.$$

In particular if we assume that no error has been committed initially (i.e. $e_0 = 0$), we have that

$$|e_n| \leq \frac{1}{4}\pi\left(e^{x_n} - 1\right)h, \quad n = 0, \ldots, N.$$

Given a positive tolerance TOL we can ensure that the error between the (unknown) analytical solution and its numerical approximation does not exceed this tolerance by choosing a positive step size h such that

$$h \leq \frac{4}{\pi}(e^{X_M} - 1)^{-1}\,\mathtt{TOL}.$$

For such h we shall have $|y(x_n) - y_n| = |e_n| \leq \mathtt{TOL}$ for each $n = 0, \ldots, N$, as required.

Definition

A general one-step method is said to be **consistent** with the ODE $y' = f(x, y)$ if the associated consistency error T_n is such that for any $\varepsilon > 0$ there exists a positive $h(\varepsilon)$ for which $|T_n| < \varepsilon$ for $0 < h < h(\varepsilon)$ and any pair of points $(x_n, y(x_n))$, $(x_{n+1}, y(x_{n+1}))$ on any solution curve contained in R.

As we have assumed that the function $\Phi(\cdot, \cdot; \cdot)$ is continuous, and also y' is a continuous function of x on $[x_0, X_M]$, it follows that

$$\lim_{h \to 0,\, x_n \to x \in [x_0, X_M]} T_n = y'(x) - \Phi(x, y(x); 0).$$

So, a general explicit one-step method is consistent if, and only if,

$$\Phi(x, y; 0) \equiv f(x, y).$$

Now we are ready to state a convergence theorem for the general one-step method.

Theorem

Suppose that the solution of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$ *lies in* R *as does its approximation generated from* $y_{n+1} = y_n + \Phi(x_n, y_n; h)$ *when* $h \leq h_0$. *Let* $\Phi(\cdot, \cdot; \cdot)$ *be uniformly continuous on* $\mathrm{R} \times [0, h_0]$ *and satisfy the consistency condition* $\Phi(x, y; 0) = f(x, y)$ *and the Lipschitz condition*

$$|\Phi(x, y; h) - \Phi(x, z; h)| \leq L_\Phi |y - z| \qquad \text{on } \mathrm{R} \times [0, h_0].$$

Then, if successive approximation sequences (y_n), *generated for* $x_n = x_0 + nh$, $n = 1, 2, \ldots, N$, *are obtained from this one-step method with successively smaller values of* h, *each less than* h_0, *we have convergence of the numerical solution to the solution of the initial-value problem in the sense that*

$$|y(x_n) - y_n| \to 0 \qquad \text{as } h \to 0,\ x_n \to x \in [x_0, X_M].$$

We saw earlier that for Euler's method the absolute value of the consistency error T_n is bounded above by a constant multiple of the step size h, that is

$$|T_n| \leq Kh \qquad \text{for } 0 < h \leq h_0,$$

where K is a positive constant, independent of h.

However there are other one-step methods (a class of which, called Runge–Kutta methods, will be considered below) for which we can do better.

To quantify the asymptotic rate of decay of the consistency error as the step size h tends to zero, we introduce the following definition.

Definition

The one-step method $y_{n+1} = y_n + h\,\Phi(x_n, y_n; h)$ is said to have **order of accuracy** p, if p is the largest positive integer such that, for any sufficiently smooth solution curve $(x, y(x))$ in R of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$, there exist constants K and h_0 such that

$$|T_n| \leq Kh^p \qquad \text{for } 0 < h \leq h_0$$

for any pair of points $(x_n, y(x_n))$, $(x_{n+1}, y(x_{n+1}))$ on the solution curve.

Next, we shall focus on a family of explicit one-step methods which have p-th order of accuracy, $p \geq 1$: explicit Runge–Kutta methods.