Numerical Solution of Differential Equations I

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Lecture 5

Carl David Tolmé Runge (30 August 1856 – 3 January 1927)
Martin Wilhelm Kutta (3 November 1867 – 25 December 1944)

Absolute stability of Runge–Kutta methods

It is instructive to consider the model problem

$$y' = \lambda y, \quad y(0) = y_0 \ (\neq 0), \tag{1}$$

with $\lambda \in \mathbb{R}_{<0}$. The analytical solution to this initial value problem,

$$y(x) = y_0 \exp(\lambda x),$$

converges to 0 at an exponential rate as $x \to +\infty$.

Question: under what conditions on the step size h does a Runge–Kutta method reproduce this behaviour?

For simplicity we restrict ourselves to the case of R-stage methods of order of accuracy R, with $1 \leq R \leq 4$.

$R = 1$

The only explicit one-stage first-order accurate Runge–Kutta method is Euler's explicit method. Applying it to (1) yields:

$$y_{n+1} = (1 + \bar{h})y_n, \quad n \geq 0,$$

where $\bar{h} := \lambda h$. Thus,

$$y_n = (1 + \bar{h})^n y_0.$$

The sequence $\{y_n\}_{n=0}^{\infty}$ will converge to 0 if, and only if,

$$|1 + \bar{h}| < 1, \qquad \text{yielding } \bar{h} \in (-2, 0).$$

For such h the explicit Euler method is said to be **absolutely stable** and the interval $(-2, 0)$ is referred to as the **interval of absolute stability** of the method.

$R = 2$

This corresponds to two-stage second-order Runge–Kutta methods:

$$y_{n+1} = y_n + h(c_1 k_1 + c_2 k_2),$$

where

$$k_1 = f(x_n, y_n), \qquad k_2 = f(x_n + a_2 h, y_n + b_{21} h k_1)$$

with

$$c_1 + c_2 = 1, \qquad a_2 c_2 = b_{21} c_2 = \frac{1}{2}.$$

Applying this to (1) yields,

$$y_{n+1} = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2\right) y_n, \qquad n \geq 0,$$

and therefore

$$y_n = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2\right)^n y_0.$$

Hence the method is absolutely stable if, and only if,

$$\left|1 + \bar{h} + \frac{1}{2}\bar{h}^2\right| < 1, \qquad \text{i.e. when } \bar{h} \in (-2, 0).$$

$R = 3$

An analogous argument shows that

$$y_{n+1} = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3\right) y_n.$$

Demanding that

$$\left|1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3\right| < 1$$

then yields the interval of absolute stability: $\bar{h} \in (-2.51, 0)$.

$R = 4$

We have that

$$y_{n+1} = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3 + \frac{1}{24}\bar{h}^4\right) y_n,$$

and the associated interval of absolute stability is $\bar{h} \in (-2.78, 0)$.

$R \geq 5$

By applying the Runge–Kutta method to the model problem (1) still results in a recursion of the form

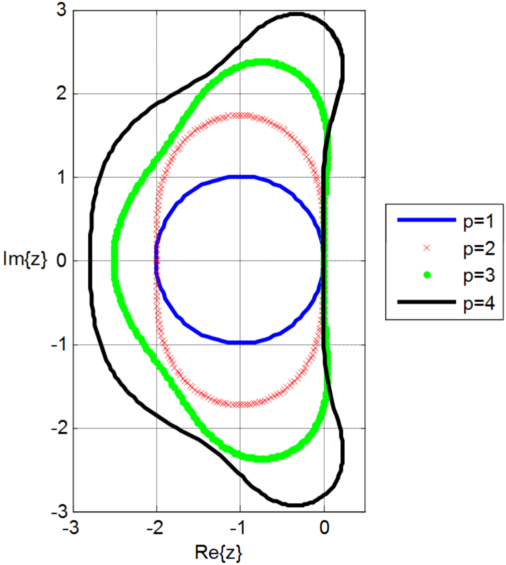
$$y_{n+1} = A_R(\bar{h})y_n, \qquad n \geq 0.$$

However, unlike the case when $R = 1, 2, 3, 4$, in addition to $\bar{h}$ now $A_R(\bar{h})$ also depends on the coefficients of the Runge–Kutta method.

By a convenient choice of the free parameters the associated interval of absolute stability may be maximised.

Regions of absolute stability of RK methods plotted in the complex plane

Consider $y' = \lambda y$, $y(0) = y_0 (\neq 0)$, with $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$.



Linear multi-step methods

While Runge–Kutta methods present an improvement over Euler's method in terms of accuracy, this comes at added computational cost, which may be more excessive than seems necessary.

Example

The fourth-order method involves 4 function evaluations per step. For comparison, by considering three consecutive points x_{n-1}, $x_n = x_{n-1} + h$, $x_{n+1} = x_{n-1} + 2h$, integrating the differential equation between x_{n-1} and x_{n+1}, and using Simpson's rule gives

$$
\begin{aligned}
y(x_{n+1}) &= y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x, y(x))\,\mathrm{d}x \\
&\approx y(x_{n-1}) + \frac{1}{3}h\left[f(x_{n-1}, y(x_{n-1})) + 4f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right],
\end{aligned}
$$

which leads to the method

$$
y_{n+1} = y_{n-1} + \frac{1}{3}h\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right].
$$

In contrast with one-step methods, where only a single value y_n was needed to compute the next approximation y_{n+1}, here we need *two* preceding values, y_n and y_{n-1} to be able to calculate y_{n+1}, and therefore the method in the last example is not a one-step method.

This is an example of a **linear multi-step method**.

Given a sequence of equally spaced mesh points (x_n) with step size h, we consider the general **linear k-step method**

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}), \tag{2}$$

where the coefficients $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_k$ are real constants. In order to avoid degenerate cases, we shall assume that $\alpha_k \neq 0$ and that α_0 and β_0 are not both equal to zero.

If $\beta_k = 0$ then y_{n+k} is obtained explicitly from previous values of y_j and $f(x_j, y_j)$, and the k-step method is then said to be **explicit**.

If $\beta_k \neq 0$ then y_{n+k} appears on both sides; because of this implicit dependence on y_{n+k} the method is then called **implicit**.

The numerical method (2) is called *linear* because it involves only linear combinations of the $\{y_n\}$ and the $\{f(x_n, y_n)\}$; for simplicity, we shall write f_n instead of $f(x_n, y_n)$.

Example

a) Euler's method is a trivial case: it is an explicit linear one-step method. The **implicit Euler method**

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

is an implicit linear one-step method.

b) The **trapezium method**

$$y_{n+1} = y_n + \frac{1}{2}h[f_{n+1} + f_n]$$

is also an implicit linear one-step method.

c) The four-step **Adams–Bashforth method**

$$y_{n+4} = y_{n+3} + \frac{1}{24}h[55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n]$$

is an explicit linear four-step method.