Numerical Solution of Differential Equations I

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Lecture 8

Absolute stability of linear multistep methods

We discussed the stability/accuracy properties of linear multistep methods in the limit of $h \to 0$, $n \to \infty$, nh fixed.

It is of practical significance to understand the performance of methods for $h > 0$ fixed and $n \to \infty$.

We must ensure that, when applied to an initial-value problem whose solution decays to zero as $x \to \infty$, the linear multistep method has a similar behaviour for $h > 0$ fixed, $x_n = x_0 + nh \to \infty$.

Our model problem with exponentially decaying solution is

$$y' = \lambda y, \quad x > 0, \qquad y(0) = y_0 \ (\neq 0), \tag{1}$$

where $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda < 0$. Indeed,

$$y(x) = y_0 \mathrm{e}^{\imath x \operatorname{Im}\lambda} \mathrm{e}^{x \operatorname{Re}\lambda},$$

and therefore,

$$|y(x)| \leq |y_0| \exp(-x|\operatorname{Re}\lambda|), \qquad x \geq 0,$$

yielding $\lim_{x\to\infty} y(x) = 0$.

Remark

We shall assume for simplicity that $\lambda \in \mathbb{R}_{<0}$, but everything extends straightforwardly to the case of $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$.

By applying a linear k-step method to the model problem (1) with $\lambda \in \mathbb{R}_{<0}$, we have:

$$\sum_{j=0}^{k} (\alpha_j - h\lambda\beta_j)\, y_{n+j} = 0.$$

The general solution y_n to this homogeneous difference equation can be expressed as a linear combination of powers of roots of the associated characteristic polynomial

$$\pi(z; \bar{h}) = \rho(z) - \bar{h}\sigma(z), \qquad (\bar{h} = h\lambda). \tag{2}$$

Thus it follows that y_n will converge to zero for $h > 0$ fixed and $n \to \infty$ if, and only if, all roots of $\pi(z; \bar{h})$ have modulus < 1.

The kth degree polynomial $\pi(z; \bar{h})$ defined by (2) is called the **stability polynomial** of the linear k-step method with first and second characteristic polynomials $\rho(z)$ and $\sigma(z)$, respectively.

Definition

A linear multistep method is called **absolutely stable** for a given $\bar{h}$ if, and only if, for that $\bar{h}$ all the roots $r_s = r_s(\bar{h})$ of the stability polynomial $\pi(z, \bar{h})$ defined by (2) satisfy $|r_s| < 1$, $s = 1, \ldots, k$. Otherwise, the method is said to be **absolutely unstable**.

An interval (α, β) of the real line is called the **interval of absolute stability** if the method is absolutely stable for all $\bar{h} \in (\alpha, \beta)$. If the method is absolutely unstable for all $\bar{h}$, it is said to have **no interval of absolute stability**.

Since for $\lambda > 0$ the solution of (1) exhibits exponential growth, it is reasonable to expect that a consistent and zero-stable (and, therefore, convergent) linear multistep method will have a similar behaviour for $h > 0$ sufficiently small, and will be therefore absolutely unstable for small $\bar{h} = \lambda h$. This is indeed the case.

Theorem

Every consistent and zero-stable linear multistep method is absolutely unstable for small positive $\bar{h}$.

Proof:

Because the method is consistent, there exists an integer $p \geq 1$ such that $C_0 = C_1 = \cdots = C_p = 0$ and $C_{p+1} \neq 0$. Consider

$$\begin{aligned}
\pi(\mathrm{e}^{\bar{h}}; \bar{h}) &= \rho(\mathrm{e}^{\bar{h}}) - \bar{h}\sigma(\mathrm{e}^{\bar{h}}) = \sum_{j=0}^{k} \left[\alpha_j \mathrm{e}^{\bar{h}j} - \bar{h}\beta_j \mathrm{e}^{\bar{h}j}\right] \\
&= \sum_{j=0}^{k} \left[\alpha_j \sum_{q=0}^{\infty} \frac{(\bar{h}j)^q}{q!} - \beta_j \sum_{q=0}^{\infty} \frac{\bar{h}^{q+1} j^q}{q!}\right] \\
&= \sum_{j=0}^{k} \left[\alpha_j \sum_{q=0}^{\infty} \frac{(\bar{h}j)^q}{q!} - \beta_j \sum_{q=1}^{\infty} \frac{\bar{h}^{q} j^{q-1}}{(q-1)!}\right] \\
&= \sum_{j=0}^{k} \alpha_j + \sum_{j=0}^{k} \left[\alpha_j \sum_{q=1}^{\infty} \frac{(\bar{h}j)^q}{q!} - \beta_j \sum_{q=1}^{\infty} \frac{\bar{h}^{q} j^{q-1}}{(q-1)!}\right] \\
&= \sum_{j=0}^{k} \alpha_j + \sum_{q=1}^{\infty} \bar{h}^q \left[\sum_{j=0}^{k} \alpha_j \frac{j^q}{q!} - \sum_{j=0}^{k} \beta_j \frac{j^{q-1}}{(q-1)!}\right] \\
&= C_0 + \sum_{q=1}^{\infty} \bar{h}^q C_q = \sum_{q=p+1}^{\infty} C_q \bar{h}^q = \mathcal{O}(\bar{h}^{p+1}). \qquad (3)
\end{aligned}$$

On the other hand, the polynomial $\pi(z;\bar{h})$ can be written in the factorised form

$$\pi(z,\bar{h}) = (\alpha_k - \bar{h}\beta_k)(z - r_1)\cdots(z - r_k)$$

where $r_s = r_s(\bar{h})$, $s = 1,\ldots,k$, are the roots of $\pi(\cdot;\bar{h})$. Hence,

$$\pi(\mathrm{e}^{\bar{h}};\bar{h}) = (\alpha_k - \bar{h}\beta_k)(\mathrm{e}^{\bar{h}} - r_1(\bar{h}))\cdots(\mathrm{e}^{\bar{h}} - r_k(\bar{h})). \qquad (4)$$

As $\bar{h} \to 0$, we have $\alpha_k - \bar{h}\beta_k \to \alpha_k \neq 0$, and thanks to the continuous dependence of the roots of a polynomial on the coefficients of the polynomial,

$$\alpha_k - \bar{h}\beta_k \to \alpha_k \neq 0 \text{ and } r_s(\bar{h}) \to \zeta_s,\ s = 1,\ldots,k,$$

where ζ_s, $s = 1,\ldots,k$, are the roots of $\rho(z)$.

Since, by assumption, the method is consistent, 1 is a root of $\rho(z)$; furthermore, by zero-stability 1 is a simple root of $\rho(z)$.

Suppose for definiteness that it is ζ_1 that is equal to 1. Then, $\zeta_s \neq 1$ for $s \neq 1$ and therefore

$$\lim_{\bar{h}\to 0}(\mathrm{e}^{\bar{h}} - r_s(\bar{h})) = (1-\zeta_s) \neq 0, \qquad s \neq 1.$$

Thus, by (4), the only factor of $\pi(\mathrm{e}^{\bar{h}};\bar{h})$ that converges to 0 as $\bar{h} \to 0$ is $\mathrm{e}^{\bar{h}} - r_1(\bar{h})$ (the other factors tend to nonzero constants).

Now, by (3), $\pi(\mathrm{e}^{\bar{h}};\bar{h}) = \mathcal{O}(\bar{h}^{p+1})$, so it follows that

$$\mathrm{e}^{\bar{h}} - r_1(\bar{h}) = \mathcal{O}(\bar{h}^{p+1}).$$

Thus we have shown that

$$r_1(\bar{h}) = \mathrm{e}^{\bar{h}} + \mathcal{O}(\bar{h}^{p+1}).$$

Hence

$$r_1(\bar{h}) > 1 + \frac{1}{2}\bar{h} \quad \text{for small positive } \bar{h}. \qquad \diamond$$

How to locate the interval of absolute stability?

We describe two methods for finding the endpoints of the interval of absolute stability.

Issai Schur
(10 January 1875, Mogilev, Belarus – 10 January 1941, Tel Aviv, Israel)

How to locate the interval of absolute stability?

We describe two methods for finding the endpoints of the interval of absolute stability.

The Schur criterion. A polynomial

$$\phi(r) = c_k r^k + \cdots + c_1 r + c_0, \qquad c_k \neq 0, \quad c_0 \neq 0,$$

with complex coefficients is said to be a **Schur polynomial** if each of its roots, r_s, satisfies $|r_s| < 1$, $s = 1, \ldots, k$.

Let

$$\hat{\phi}(r) := \bar{c}_0 r^k + \bar{c}_1 r^{k-1} + \cdots + \bar{c}_{k-1} r + \bar{c}_k,$$

where $\bar{c}_j$ denotes the complex conjugate of c_j, $j = 1, \ldots, k$.

Further, let us define

$$\phi_1(r) = \frac{1}{r}\left[\hat{\phi}(0)\phi(r) - \phi(0)\hat{\phi}(r)\right].$$

Clearly ϕ_1 has degree $\leq k - 1$.

The following key result is stated without proof.

Theorem (Schur's criterion)

The polynomial ϕ is a Schur polynomial if, and only if:

- $|\hat{\phi}(0)| > |\phi(0)|$, *and*
- ϕ_1 *is a Schur polynomial.*

Exercise

Use Schur's criterion to determine the interval of absolute stability of the linear multistep method

$$y_{n+2} - y_n = \frac{h}{2}\left(f_{n+1} + 3f_n\right).$$

SOLUTION: The first and second characteristic polynomials of the method are

$$\rho(z) = z^2 - 1, \qquad \sigma(z) = \frac{1}{2}(z+3).$$

Therefore the stability polynomial is

$$\pi(r;\overline{h}) = \rho(r) - \overline{h}\sigma(r) = r^2 - \frac{1}{2}\overline{h}r - \left(1 + \frac{3}{2}\overline{h}\right).$$

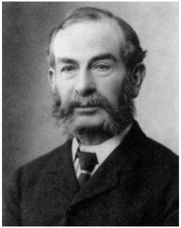
Now,

$$\hat{\pi}(r;\overline{h}) = -\left(1 + \frac{3}{2}\overline{h}\right) r^2 - \frac{1}{2}\overline{h}r + 1.$$

Clearly, $|\hat{\pi}(0;\overline{h})| > |\pi(0,\overline{h})|$ if, and only if, $\overline{h} \in (-\frac{4}{3}, 0)$. As

$$\pi_1(r,\hat{h}) = -\frac{1}{2}\overline{h}(2 + \frac{3}{2}\overline{h})(3r+1)$$

has the unique root $-\frac{1}{3}$ and is, therefore, a Schur polynomial, we deduce from Schur's criterion that $\pi(r;\overline{h})$ is a Schur polynomial if, and only if, $\overline{h} \in (-\frac{4}{3}, 0)$. Therefore the interval of absolute stability is $(-\frac{4}{3}, 0)$. $\diamond$



Edward John Routh

20 January 1831, Quebec – 7 June 1907 Cambridge

Adolf Hurwitz

26 March 1859 Hildesheim – 18 November 1919 Zürich

The Routh–Hurwitz criterion. Consider the mapping

$$z = \frac{r-1}{r+1}$$

of the open unit disc $|r| < 1$ of the complex r-plane to the left open complex half-plane $\operatorname{Re} z < 0$ of the complex z-plane.

The inverse of this mapping is

$$r = \frac{1+z}{1-z}.$$

Under this transformation $\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r)$ becomes

$$\rho\left(\frac{1+z}{1-z}\right) - \bar{h}\sigma\left(\frac{1+z}{1-z}\right).$$

Multiplying this by $(1-z)^k$ we obtain a polynomial of the form

$$a_0 z^k + a_1 z^{k-1} + \cdots + a_k. \tag{5}$$

The roots of $\pi(r, \bar{h})$ lie inside the open unit disk $|r| < 1$ if, and only if, the roots of (5) lie in the left open complex half-plane $\operatorname{Re} z < 0$.

Theorem (Routh–Hurwitz criterion)

The roots of (5) lie in the left open complex half-plane if, and only if, all the leading principal minors of the $k \times k$ matrix

$$Q = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & a_0 & a_2 & \cdots & a_{2k-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix}$$

are positive and $a_0 > 0$; we assume that $a_j = 0$ if $j > k$. E.g.:

a) *for $k = 2$: $a_0 > 0$, $a_1 > 0$, $a_2 > 0$;*

b) *for $k = 3$: $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 a_0 > 0$;*

c) *for $k = 4$: $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, $a_1 a_2 a_3 - a_0 a_3^2 - a_4 a_1^2 > 0$;*

represent the necessary and sufficient conditions for ensuring that all roots of (5) lie in the left open complex half-plane.

Exercise

Use the Routh–Hurwitz criterion to find the interval of absolute stability of the linear multistep method from the previous exercise.

SOLUTION: By applying the substitution

$$r = \frac{1+z}{1-z}$$

in the stability polynomial

$$\pi(r, \bar{h}) = r^2 - \frac{1}{2}\bar{h}r - \left(1 + \frac{3}{2}\bar{h}\right)$$

and multiplying the resulting function by $(1-z)^2$, we get

$$(1-z)^2 \left[\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{2}\bar{h}\left(\frac{1+z}{1-z}\right) - \left(1 + \frac{3}{2}\bar{h}\right)\right] = a_0 z^2 + a_1 z + a_2$$

with

$$a_0 = -\bar{h}, \qquad a_1 = 4 + 3\bar{h}, \qquad a_2 = -2\bar{h}.$$

Applying part a) of the theorem (Routh–Hurwitz criterion) we deduce that the method is zero-stable if, and only if, $\bar{h} \in (-\frac{4}{3}, 0)$; hence the interval of absolute stability is $(-\frac{4}{3}, 0)$. $\diamond$