Numerical Solution of Differential Equations I

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 12

Finite difference approximation of parabolic equations

As a simple but representative model problem we focus on the unsteady diffusion equation (heat equation) in one space dimension:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

which we shall consider for $x \in (-\infty, \infty)$ and $t \geq 0$, subject to the initial condition

$$u(x, 0) = u_0(x), \qquad x \in (-\infty, \infty),$$

where u_0 is a given function.

The solution of this initial-value problem can be expressed explicitly in terms of the initial datum u_0.

We summarize here the derivation of this expression.

We recall that the Fourier transform of a function v is defined by

$$\hat{v}(\xi) = F[v](\xi) = \int_{-\infty}^{\infty} v(x)\, \mathrm{e}^{-\imath x\xi}\, \mathrm{d}x.$$

We shall assume henceforth that the functions under consideration are sufficiently smooth and that they decay to $\pm\infty$ sufficiently quickly in order to ensure that our manipulations make sense.

By Fourier-transforming the PDE (1) we obtain

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\, \mathrm{e}^{-\imath x\xi}\, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)\, \mathrm{e}^{-\imath x\xi}\, \mathrm{d}x.$$

After (formal) integration by parts on the right-hand side and ignoring boundary terms at $\pm\infty$, we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = (\imath\xi)^2\hat{u}(\xi, t),$$

whereby

$$\hat{u}(\xi, t) = \mathrm{e}^{-t\xi^2}\hat{u}(\xi, 0),$$

and therefore

$$u(x, t) = F^{-1}\left(\mathrm{e}^{-t\xi^2}\hat{u}_0\right).$$

The inverse Fourier transform of a function is defined by

$$v(x) = F^{-1}[\hat{v}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}(\xi)\mathrm{e}^{\imath x\xi}\,\mathrm{d}\xi.$$

After some lengthy calculations, which we omit, we find that

$$u(x,t) = F^{-1}\left(\mathrm{e}^{-t\xi^2}\hat{u}_0(\xi)\right) = \int_{-\infty}^{\infty} w(x-y,t)u_0(y)\,\mathrm{d}y,$$

where the function w, defined by

$$w(x,t) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-x^2/(4t)},$$

is called the **heat kernel**. So, finally,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2/(4t)}u_0(y)\,\mathrm{d}y. \tag{2}$$

This formula gives an explicit expression of the solution of the heat equation (1) in terms of the initial datum u_0. Because $w(x,t) > 0$ for all $x \in (-\infty, \infty)$ and all $t > 0$, and

$$\int_{-\infty}^{\infty} w(y,t)\,\mathrm{d}y = 1 \qquad \text{for all } t > 0,$$

we deduce from (2) that if u_0 is a bounded continuous function, then

$$\sup_{x\in(-\infty,+\infty)}|u(x,t)| \leq \sup_{x\in(-\infty,\infty)}|u_0(x)|, \qquad t > 0. \tag{3}$$

In other words, the 'largest' and 'smallest' values of $u(\cdot,t)$ at $t > 0$ cannot exceed those of $u_0(\cdot)$.

Similar bounds on the 'magnitude' of the solution at future times in terms of the 'magnitude' of the initial datum can be obtained in other norms as well, and we shall focus here on the L^2 norm.

We will show, using Parseval's identity, that the L^2 norm of the solution, at any time $t > 0$, is bounded by the L^2 norm of the initial datum.

We shall then try to mimic this when using various numerical approximations of the initial-value problem for the heat equation.

Lemma (Parseval's identity)

Suppose that $u \in L^2(-\infty,\infty)$*. Then,* $\hat{u} \in L^2(-\infty,\infty)$*, and the following equality holds:*

$$\|u\|_{L^2(-\infty,\infty)} = \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L^2(-\infty,\infty)},$$

where

$$\|u\|_{L^2(-\infty,\infty)} = \left(\int_{-\infty}^{\infty} |u(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

PROOF. We begin by observing that

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{u}(\xi)\, v(\xi)\, \mathrm{d}\xi &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(x)\, \mathrm{e}^{-\imath x\xi}\, \mathrm{d}x \right) v(\xi)\, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(\xi)\, \mathrm{e}^{-\imath x\xi}\, \mathrm{d}\xi \right) u(x)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} u(x)\, \hat{v}(x)\, \mathrm{d}x.\end{aligned}$$

We then take

$$v(\xi) = \overline{\hat{u}(\xi)} = 2\pi F^{-1}[\bar{u}](\xi)$$

and substitute this into the identity above. $\diamond$

Returning to equation (1), we thus have by Parseval's identity that

$$\|u(\cdot, t)\|_{L^2(-\infty,\infty)} = \frac{1}{\sqrt{2\pi}} \|\hat{u}(\cdot, t)\|_{L^2(-\infty,\infty)}, \qquad t > 0.$$

Therefore,

$$\begin{aligned} \|u(\cdot, t)\|_{L^2(-\infty,\infty)} &= \frac{1}{\sqrt{2\pi}} \|\mathrm{e}^{-t\xi^2} \hat{u}_0(\cdot)\|_{L^2(-\infty,\infty)} \\ &\leq \frac{1}{\sqrt{2\pi}} \|\hat{u}_0\|_{L^2(-\infty,\infty)} \\ &= \|u_0\|_{L^2(-\infty,\infty)}, \qquad t > 0. \end{aligned}$$

Thus we have shown that

$$\|u(\cdot, t)\|_{L^2(-\infty,\infty)} \leq \|u_0\|_{L^2(-\infty,\infty)} \qquad \text{for all } t > 0. \tag{4}$$

This is a useful result as it can be used to deduce stability of the solution of the equation (1) with respect to perturbations of the initial datum in a sense which we shall now explain.

Suppose that u_0 and $\tilde{u}_0$ are two functions contained in $L^2(-\infty, \infty)$ and denote by u and $\tilde{u}$ the solutions to (1) resulting from the initial functions u_0 and $\tilde{u}_0$, respectively.

Then $u - \tilde{u}$ solves the heat equation with initial datum $u_0 - \tilde{u}_0$, and therefore, by (4), we have that

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^2(-\infty,\infty)} \leq \|u_0 - \tilde{u}_0\|_{L^2(-\infty,\infty)} \qquad \text{for all } t > 0.$$

This inequality implies continuous dependence of the solution on the initial function: small perturbations in u_0 in the $L^2(-\infty, \infty)$ norm will result in small perturbations in the associated analytical solution $u(\cdot, t)$ in the $L^2(-\infty, \infty)$ norm for all $t > 0$.

Inequality (4) is therefore a relevant property, which we shall try to mimic with our numerical approximations of the equation (1).

Analogously,

$$\sup_{x \in (-\infty, \infty)} |u(x,t) - \tilde{u}(x,t)| \leq \sup_{x \in (-\infty, \infty)} |u_0(x) - \tilde{u}_0(x)| \qquad \text{for all } t > 0.$$